

The Riemann problem of the Burgers equation with a discontinuous source term

Beixiang Fang, Pingfan Tang*, Ya-Guang Wang

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China

ARTICLE INFO

Article history:

Received 26 December 2011

Available online 17 May 2012

Submitted by Dehua Wang

Keywords:

Riemann problems

Burgers equation

Discontinuous source terms

“vacuum” regions

Curved shocks and rarefaction waves

Weak discontinuities

Appearance and disappearance of new shocks

Entropy solutions

ABSTRACT

In this paper we are concerned with the Riemann problem of the Burgers equation with a discontinuous source term, motivated by studying the propagation of singular waves in radiation hydrodynamics. By calculating the representation of solutions, we construct the global entropy solution to this Riemann problem, in which one needs to pay attention to the effects of the discontinuous source term on the propagation of the Riemann waves. It turns out that the discontinuity of the source term has clear influences on the shock or rarefaction waves generated by the initial Riemann data, and produces some new and interesting phenomena such as the appearance of weak discontinuities, the appearance and absorption of new shocks, artificial “vacuums”, and different types of asymptotic behavior of shocks. These new waves and phenomena shall be analyzed in this paper.

© 2012 Elsevier Inc. All rights reserved.

Contents

1. Introduction.....	307
2. The case of the initial data generating a shock	309
2.1. Propagation of a shock without vacuum.....	309
2.2. Propagation of a shock with vacuum and a possible bifurcation	318
3. The case of the initial data generating a rarefaction wave	323
3.1. Bend of a rarefaction wave without shock.....	323
3.2. Bend of a rarefaction wave with a shock.....	329
4. Uniqueness and stability of entropy solutions.....	331
Acknowledgments	335
References.....	335

1. Introduction

In this paper, we are concerned with the Riemann problem of the Burgers equation with a discontinuous source term. There are quite a number of physical phenomena that can be described by equations of hyperbolic conservation laws with source terms. The radiation hydrodynamics model, which is formulated as gas dynamic equations coupled with a nonlocal transport equation for the radiation field (cf. [1,2]) is a remarkable example. In certain situations, the action of the radiation fields on flow can be reformulated as source terms for the gas dynamic equations. It has been shown by Zhong and Jiang

* Corresponding author.

E-mail addresses: bxfang@sjtu.edu.cn (B. Fang), tangpingfan@sjtu.edu.cn (P. Tang), ygwang@sjtu.edu.cn (Y.-G. Wang).

in [3] that, in general, the smooth solution to such system will blow up in finite time (also see [4] for a proof on formation of singularities). Therefore, to understand the behavior of singular waves in radiative hydrodynamics, it is important and also a starting point to study the gas dynamic system with discontinuous sources. In the study of transonic nozzle flow, one also meets problems to study the weak solutions of conservation laws with discontinuous source terms, e.g. cf. [5] and references therein.

Some interesting works have been done for the equations of conservation laws with (possibly discontinuous) source terms, for instance, see [6–9,5,10]. However, these works mainly concern the global BV solutions and/or asymptotic behavior of weak solutions, while, to our knowledge, the structure of the nonlinear waves in these equations has not been studied yet. Both in theory and applications, it is valuable and important to describe the structural behavior of nonlinear waves in hyperbolic conservation laws with discontinuous sources. As an important model, in this paper we consider the following Riemann problem of the Burgers equation with a discontinuous source term,

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = g(x, t) & x \in \mathbb{R}, t > 0, \\ u(x, 0) = \begin{cases} u_0^-, & x < 0 \\ u_0^+, & x \geq 0 \end{cases} \end{cases} \quad (1.1)$$

where u_0^- and u_0^+ are two constants, and $g(x, t) = g^+ H(x) + g^- H(-x)$ with $g^+, g^- \in \mathbb{R}$ being two constants, and $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ the Heaviside function.

For the Riemann problem of the Burgers equation without the source term:

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0 \quad (1.2)$$

it is well known that a shock or a rarefaction wave will be generated [11,12]. Then as the (discontinuous) source term $g(x, t)$ is added to the Eq. (1.2), it will act on the shock or the rarefaction wave. In this paper, we shall focus on the new phenomena caused by the actions of the discontinuity of the source term. Definitely, this study shall give us valuable insight on the behavior of entropy solutions to the equations of radiation hydrodynamics. It is worthy of noting that the argument in this paper for the problem (1.1) can be extended to study local structure of entropy solutions to equation of scalar conservation law with general discontinuous source terms.

In the case that the initial Riemann data generate a shock for the Eq. (1.2), the source term $g(x, t)$ has an impact on the strength of the shock front and there are two interesting phenomena due to the discontinuity of source. One relates to the appearance and disappearance of a new shock with respect to the ratio $\frac{g^+}{g^-}$ when g^+ and g^- are positive. Indeed, suppose $u_0^+ < u_0^- < 0$, that is, the initial data generate a shock, which moves into the second quadrant. Then if g^+ and g^- are positive, the action of the source term yields three different patterns of entropy solutions as the ratio $\frac{g^+}{g^-}$ increases. When this ratio is small such that $0 < \frac{g^+}{g^-} \leq \frac{1}{2}$, the action of the discontinuous source term curves the main shock front to the right, then crossing the discontinuous line $\{x = 0\}$ of the source and entering the first quadrant, and the solution has a weak discontinuity on $\{x = 0\}$. Then as $\frac{g^+}{g^-}$ increases over a critical value such that $\frac{u_0^+}{u_0^- + u_0^+} < \frac{g^+}{g^-} < 1$, the main shock is still bent to the right and enters the first quadrant, but because of the compression of the characteristics, a new shock forms from a point on the discontinuous line $\{x = 0\}$ of the source, propagates to the right and then interacts with that main shock. However, as $\frac{g^+}{g^-}$ continues to increase such that $\frac{g^+}{g^-} \geq 1$, there will be no new shock anymore; instead, a weak discontinuity appears below the main shock front. We will describe in detail these three patterns of entropy solutions in Section 2.1. The other interesting new phenomena is that when $g^- < 0 < g^+$, the source term may create “vacuum”, a region in t - x plane where the solution cannot be determined directly from the initial data by integrating along characteristics. We can determine the solution in this “vacuum” region by an approximation argument for the data, and this solution also satisfies the entropy condition. Moreover, it turns out that there are different, complicated shock solution patterns for this case. We shall establish four patterns in detail in Section 2.2 with respect to the assumption that $u_0^+ < 0 < u_0^-, u_0^- + u_0^+ < 0, g^- < 0 < g^+$ and $\frac{u_0^-}{g^-} < \frac{u_0^+}{g^+}$.

In the case that the initial Riemann data generate a rarefaction wave, the action of the discontinuous source term is similar to the case of shock. According to different values of the pair (g^-, g^+) , the source term may create a weak discontinuity, a shock front or a vacuum. We shall study the entropy solutions of this case in Section 3. First in Section 3.1, we consider the case that the rarefaction wave is bent by the source term and also the case of appearing a “vacuum” region. We determine the solution in this “vacuum” region as the limit of an approximate solution sequence to the problems with the discontinuous source term being approximated by a continuous one. This approach makes sense because the limit is independent of the choice of the approximating sequence of the source term. This will be shown by proving the uniqueness and stability of the entropy solution to the problem (1.1) with respect to the initial data and the source term in Section 4. In Section 3.2, we will construct solutions with both rarefaction waves and shock waves, generated by the action of discontinuous source terms.

2. The case of the initial data generating a shock

Without loss of generality, in Sections 2 and 3, we assume that the source term satisfies $g^- + g^+ > 0$. This can be realized by applying the following transformation,

$$\tilde{u}(x, t) = -u(-x, t) \quad \tilde{g}(x, t) = -g(-x, t)$$

as $g^- + g^+ < 0$.

In this section, we consider the case that the initial Riemann data generate a shock and investigate the action of the discontinuous source term on the shock. We employ the characteristic methods to construct the entropy solution showing the basic action of the source term: curving the shock front and changing its strength. Then we focus on two interesting phenomena. One relates to the appearance and disappearance of a new shock, and the other is the appearance of a “vacuum” region. We shall construct entropy solutions corresponding to these phenomena.

2.1. Propagation of a shock without vacuum

In this subsection, we will first show that the discontinuous source term can curve the shock front and make the strength of the shock increase to infinity or diminish to zero as t tends to infinity, depending on whether $\frac{g^+}{g^-}$ is larger or smaller than one.

Secondly, by constructing the solutions through integration along characteristics, we observe the phenomena of the appearance and disappearance of a new shock determined by the ratio $\frac{g^+}{g^-}$. More precisely, assume that g^+ and g^- are positive, and $u_0^+ < u_0^- < 0$, i.e. the shock front generated by the initial data moves into the second quadrant. When the ratio $\frac{g^+}{g^-}$ is smaller or equal to $\frac{1}{2}$, the discontinuous source term just bends the main shock and creates a weak discontinuity on $\{x = 0\}$. This pattern of solution is described in Theorem 2.3. As $\frac{g^+}{g^-}$ increases over a critical value but smaller than one, there appears a new shock front issuing from one point on $\{x = 0\}$ and interacting with the main shock. This solution is described in Theorems 2.4 and 2.5. While $\frac{g^+}{g^-}$ continues to increase such that $\frac{g^+}{g^-} \geq 1$, there will be no new shock anymore. Instead, a new weak discontinuity appears in the front of the main shock front. This phenomenon is illustrated in Remark 2.2.

In the following theorem, the shock enters the first quadrant from the origin and the discontinuous sources curve the front. The proof of this theorem also presents the method of characteristics mainly we used in this paper. For simplicity of presentation, we define

$$u_3(x, t) = \begin{cases} g^+t + u_0^- + \frac{g^- - g^+}{g^+ - 2g^-} \left(g^+t - g^-t + u_0^- - \sqrt{(g^-t + u_0^-)^2 + 2x(g^+ - 2g^-)} \right) & g^+ \neq 2g^- \\ g^-t + u_0^- + \frac{g^-x}{g^-t + u_0^-} & g^+ = 2g^- \end{cases} \quad (2.1)$$

Theorem 2.1. Suppose that $u_0^- + u_0^+ > 0$, and $g^- > 0$, then the problem (1.1) has a piecewise C^1 solution $u(x, t)$,

$$u(x, t) = \begin{cases} u_1(x, t) = g^-t + u_0^-, & (x, t) \in R_1^1 \\ u_2(x, t) = g^+t + u_0^+, & (x, t) \in R_2^1 \\ u_3(x, t), & (x, t) \in R_3^1 \end{cases} \quad (2.2)$$

where u_3 is given in (2.1), and the regions R_i^1 ($i = 1, 2, 3$) are given by

$$R_1^1 = \{x < 0, t \geq 0\}$$

$$R_2^1 = \{x > \varphi(t), t \geq 0\}$$

$$R_3^1 = \{0 < x < \varphi(t), t \geq 0\}$$

with $x = \varphi(t)$ being the front determined by

$$\begin{cases} \varphi'(t) = \frac{1}{2}(u_3(\varphi(t), t) + u_2(\varphi(t), t)), & t > 0 \\ \varphi(0) = 0. \end{cases} \quad (2.3)$$

In the solution (2.2), (u_3, u_2, φ) is a shock with $x = \varphi(t)$ being its front and $\{x = 0\}$ is a weak singularity of $u(x, t)$, i.e. u_1, u_3 are continuous at $\{x = 0\}$, while their derivatives have jump.

Proof. We construct the solution by the method of characteristics. For the problem (1.1), the solution $u(x, t)$ can be represented as

$$\begin{cases} \frac{du}{dt} = g(x(t), t) \\ u(x(t_0), t_0) = u_0 \end{cases}$$

with $\{x = x(t)\}$ given by

$$\begin{cases} \frac{dx}{dt} = u(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

being the characteristic line passing through (x_0, t_0) .

Since

$$g(x, t) = \begin{cases} g^- & x < 0 \\ g^+ & x > 0 \end{cases}$$

we have that in case $x_0 < 0$ and $x(t) < 0$, that is, the characteristic curve stays in the left part of the discontinuous line $\{x = 0\}$ of the source term $g(x, t)$, a direct computation yields

$$\begin{cases} x(t; x_0, t_0) = \frac{1}{2}g^-(t - t_0)^2 + u_0(t - t_0) + x_0 \\ u(x(t), t; x_0, t_0) = g^-(t - t_0) + u_0, \end{cases}$$

while in case $x_0 > 0$ and $x(t) > 0$, the characteristic curve stays in the right part of $\{x = 0\}$, we have

$$\begin{cases} x(t; x_0, t_0) = \frac{1}{2}g^+(t - t_0)^2 + u_0(t - t_0) + x_0 \\ u(x(t), t; x_0, t_0) = g^+(t - t_0) + u_0. \end{cases}$$

By the assumption $u_0^- + u_0^+ > 0$, and the shock front $x = \varphi(t)$ satisfying

$$\varphi'(0) = \frac{1}{2}(u_0^- + u_0^+) > 0$$

we know that the shock front $x = \varphi(t)$ enters the right part of $\{x = 0\}$ when t is small, that is, $\varphi(t) > 0$.

Thus, the characteristic curves starting from $(x_0, 0)$ on x -axis can cover the whole region $R_1^1 = \{x < 0, t \geq 0\}$, and the solution $u(x, t)$ in R_1^1 is given by:

$$u(x, t) = u_1(x, t) = g^-t + u_0^-$$

on each characteristics

$$x = x(t; x_0, 0) = \frac{1}{2}g^-t^2 + u_0^-t + x_0.$$

Then we are going to solve the boundary value problem

$$\begin{cases} u_t + \left(\frac{1}{2}u^2\right)_x = g^+ & t > 0, x > 0 \\ u(0, x) = u_0^+ & x > 0 \\ u(t, 0) = g^-t + u_0^- \end{cases} \quad (2.4)$$

By a direct computation, we see that the boundary value problem (2.4) has a shock solution $(u(x, t), \varphi(t))$

$$u(x, t) = \begin{cases} u_2(x, t) & 0 < x < \varphi(t), t > 0 \\ u_3(x, t) & x > \varphi(t), t > 0 \end{cases}$$

with $\varphi(t)$ being given in (2.3).

Under the assumptions of this theorem, one can check that

$$\varphi'(t) > 0 \quad \text{for all } t > 0.$$

Thus, the above construction gives a global shock solution to this problem. \square

Fig. 2.1 illustrates the solution (2.2) and its characteristic curves under the assumption $u_0^- > 0 > u_0^+$ and $g^+ > 0$.

Now on the basis of Theorem 2.1, we study the influence of the discontinuous source term on the strength of the shock front when t goes to infinity.

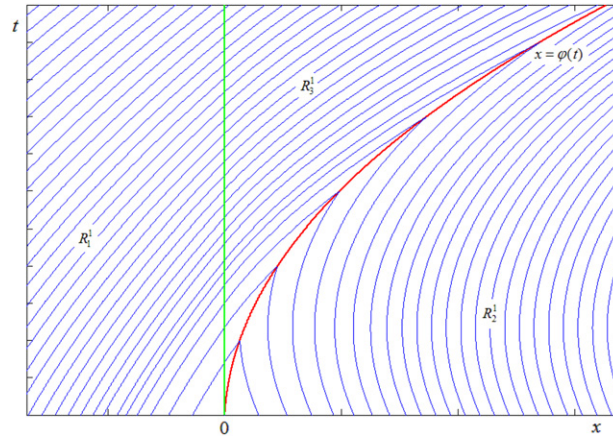


Fig. 2.1. The shock front is curved by the source term.

Theorem 2.2. Assume $u_0^- > u_0^+$, $g^- + g^+ > 0$, and $x = \varphi(t)$ is the front of a shock for problem (1.1). The strength of this shock is defined by

$$J(t) = \lim_{x \rightarrow \varphi(t)^-} u(x, t) - \lim_{x \rightarrow \varphi(t)^+} u(x, t), \quad (2.5)$$

then, we have

$$\lim_{t \rightarrow +\infty} J(t) = \begin{cases} 0, & \text{if } g^- < g^+ \\ +\infty, & \text{if } g^- > g^+. \end{cases} \quad (2.6)$$

Proof. We only prove this result under the condition $u_0^- > 0 > u_0^+$, $u_0^+ + u_0^- > 0$ and $g^- > 0$, $g^+ > 0$. The other cases can be studied similarly.

(1) If $g^+ \neq 2g^-$, from

$$J(t) = u_3(\varphi(t), t) - u_2(\varphi(t), t) \quad (2.7)$$

and (2.3), we have

$$J'(t) = \frac{g^- - g^+}{2\sqrt{(g^-t + u_0^-)^2 + 2\varphi(t)(g^+ - 2g^-)}} J(t).$$

When $g^+ < 2g^-$ and $g^+ > g^-$, we get

$$J'(t) \leq -\frac{C}{t} J(t) \quad \text{for } t \geq t_0 \quad (2.8)$$

for two positive constants C and t_0 . From (2.8), it follows immediately

$$J(t) \leq C_1 t^{-C_2} \quad \text{for } t \geq t_0 \quad (2.9)$$

where C_1 and C_2 are two positive constants. Therefore, $J(t) \rightarrow 0$ as $t \rightarrow +\infty$.

If $g^+ > 2g^-$, from (2.3) we have $\varphi'(t) \leq (g^+ + g^-)t + C$, which implies $\varphi(t) \leq C_1 t^2$ for some positive constant C_1 when t is large enough. So, (2.8) still holds and so does (2.9).

If $g^+ < g^-$, then we have

$$J'(t) \geq \frac{C}{t} J(t) \quad \text{for } t \geq t_0$$

where C and t_0 are two positive constants. By integration, we obtain when t is large enough $J(t) \geq C_1 t^{C_2}$ for two positive constants C_1 and C_2 . Therefore, $J(t) \rightarrow +\infty$ as $t \rightarrow +\infty$.

(2) If $g^+ = 2g^-$, the strength of the shock is $J(t) = u_0^- - u_0^+ - g^-t + \frac{g^-\varphi(t)}{g^-t + u_0^-}$. By a direct computation, we get

$$J'(t) = \frac{-g^-}{2(g^-t + u_0^-)} J(t), \quad J(0) = u_0^- - u_0^+$$

which implies $J(t) = \frac{(u_0^- - u_0^+) \sqrt{u_0^-}}{\sqrt{g^-t + u_0^-}} = \mathcal{O}(\frac{1}{\sqrt{t}})$ immediately, as $t \rightarrow +\infty$. \square

Remark 2.1. Theorem 2.2 shows that, as t increases, the strength of the shock wave in the solution of (1.1) may increase to infinity or diminish to zero and this is also due to the existence of the source term and its discontinuity. In fact, under the condition $u_0^- > 0 > u_0^+$, $u_0^+ + u_0^- > 0$ and $g^- > 0$, $g^+ > 0$ we can get a more precise result about the strength of the shock front:

$$J(t) = \begin{cases} \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), & \text{as } t \rightarrow +\infty \text{ if } g^- < g^+ \\ \mathcal{O}(t), & \text{as } t \rightarrow +\infty \text{ if } g^- > g^+. \end{cases} \quad (2.10)$$

This result has been proven when $g^+ = 2g^-$. If $g^+ \neq 2g^-$, let $p(t) = \varphi'(t)$ and from (2.3) we have

$$\begin{aligned} p'(t) = & \left((3(g^+)^2 - 6g^+g^- - (g^-)^2)p + (3g^+g^- - (g^+)^2)(g^+ + g^-)t \right. \\ & \left. + 2g^+g^-u_0^- - ((g^+)^2 - 2g^+g^- - (g^-)^2)u_0^+ \right) / \left(4(g^+ - 2g^-)p \right. \\ & \left. - 2((g^+)^2 - 2g^+g^- - (g^-)^2)t + 2g^-u_0^- - 2(g^+ - 2g^-)u_0^+ \right). \end{aligned} \quad (2.11)$$

By solving (2.11) we obtain

$$\left(p + \frac{-(g^+)^2 + 2g^+g^- + 3(g^-)^2}{4(g^+ - 2g^-)}t - \frac{1}{4}u_0^+ + \frac{3g^-u_0^-}{4(g^+ - 2g^-)} \right) (p - g^+t - u_0^+)^2 = C \quad (2.12)$$

where C is a constant determined by the initial data. So, from $p(t) = \varphi'(t) = \frac{1}{2}J(t) + g^+t + u_0^+$ and (2.12), we get

$$(J(t))^2 \left(2J(t) + \frac{3(g^+ - g^-)^2}{g^+ - 2g^-}t + 3u_0^+ + \frac{3g^-}{g^+ - 2g^-}u_0^- \right) = C, \quad \forall t \geq 0. \quad (2.13)$$

Combining (2.6) and (2.13), it follows (2.10) immediately.

Now, we study the first interesting phenomenon caused by the discontinuous source term: the appearance and disappearance of a new shock front as the ratio $\frac{g^+}{g^-}$ changes. If $0 > u_0^- > u_0^+$ and $g^+ + g^- > 0$, the shock front enters the second quadrant from the origin but it will move into the first quadrant as t increases. For simplicity of presentation, we define

$$u_4(x, t) = \begin{cases} g^-t + u_0^+ + \frac{g^+ - g^-}{g^- - 2g^+} \left(g^-t - g^+t + u_0^+ + \sqrt{(g^+t + u_0^+)^2 + 2x(g^- - 2g^+)} \right) & g^- \neq 2g^+ \\ g^+t + u_0^+ + \frac{g^+x}{g^+t + u_0^+} & g^- = 2g^+. \end{cases} \quad (2.14)$$

As $\frac{g^+}{g^-} \in (0, \frac{1}{2}]$, we have the following simple result.

Theorem 2.3. Suppose that $0 > u_0^- > u_0^+$, and $0 < \frac{g^+}{g^-} \leq \frac{1}{2}$, then the problem (1.1) has a piecewise \mathcal{C}^1 solution containing one shock and a weak discontinuity on $\{x = 0\}$,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^2 \quad (i = 1, 2, 3, 4) \quad (2.15)$$

where u_1, u_2, u_3, u_4 are given in (2.2), (2.2), (2.1), (2.14) respectively and the regions R_i^2 ($i = 1, 2, 3, 4$) are given by

$$\begin{aligned} R_1^2 &= \{x < x_1(t), 0 \leq t < t_p\} \cup \{x < 0, t \geq t_p\} \\ R_2^2 &= \{x > 0, 0 \leq t < t_p\} \cup \{x > x_2(t), t \geq t_p\} \\ R_3^2 &= \{0 < x < x_2(t), t \geq t_p\} \\ R_4^2 &= \{x_1(t) < x < 0, 0 \leq t < t_p\} \end{aligned}$$

where $P = (0, t_p)$ is the intersection point of $\{x = x_1(t)\}$ and t -axis, and the shock front is given by

$$x = \begin{cases} x_1(t), & \text{as } 0 \leq t \leq t_p \\ x_1(t), & \text{as } t > t_p \end{cases}$$

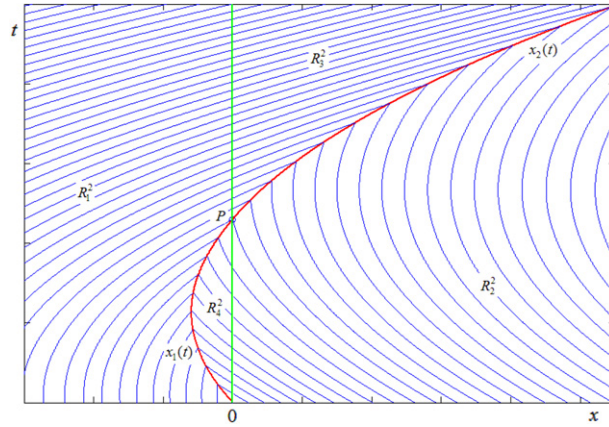


Fig. 2.2. There is only one shock front in the solution (2.15) when $\frac{g^+}{g^-} \in (0, \frac{1}{2}]$.

with $x_1(t)$ and $x_2(t)$ being given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)), & t > 0 \\ x_1(0) = 0 \end{cases}$$

and

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_3(x_2(t), t) + u_2(x_2(t), t)), & t > t_p \\ x_2(t_p) = 0. \end{cases}$$

The regions R_i^2 ($i = 1, 2, 3, 4$), curves $x_1(t)$, $x_2(t)$ and the characteristics of the solution (2.15) are shown in Fig. 2.2.

Proof. From the Rankine–Hugoniot jump conditions, we have $x_1'(0) = \frac{1}{2}(u_0^- + u_0^+)$, so $x_1'(t) < 0$ for small t , which means some characteristic curves starting from $\{x > 0, t = 0\}$ will cross the t -axis in a neighborhood of the origin.

By the method of characteristics we get the values of u_1 and u_2 immediately.

For any point $(0, t_0)$, $t_0 \in (0, t_Q)$ with $t_Q = -\frac{u_0^+}{g^+}$, we consider the characteristic curve $x = x(t; t_0)$ starting from this point and extended to the left part of t -axis:

$$\begin{cases} x'(t) = u(x(t), t) \\ x(t_0) = 0 \\ x(t) < 0 \\ u(0, t_0) = u_2(0, t_0). \end{cases} \quad (2.16)$$

By solving (2.16) we get:

$$\begin{cases} x(t; t_0) = \frac{1}{2}g^-t^2 + (u_0^+ + g^+t_0 - g^-t_0)t + \left(\frac{1}{2}g^- - g^+\right)t_0^2 - u_0^+t_0 \\ t_0 = \frac{1}{g^- - 2g^+} \left(u_0^+ - g^+t + g^-t + \sqrt{(g^+t + u_0^+)^2 + 2x(g^- - 2g^+)}\right) \\ u(x, t) = u_4(x, t). \end{cases} \quad (2.17)$$

From (2.17) we know that (x, t) should satisfy:

$$\begin{cases} 0 < t_0 = \frac{1}{g^- - 2g^+} \left(u_0^+ - g^+t + g^-t + \sqrt{(g^+t + u_0^+)^2 + 2x(g^- - 2g^+)}\right) < t_Q \\ (g^+t + u_0^+)^2 + 2x(g^- - 2g^+) > 0. \end{cases} \quad (2.18)$$

We solve (2.18) under the condition $0 > u_0^- > u_0^+$ and $0 < \frac{g^+}{g^-} < \frac{1}{2}$ and then get:

$$(x, t) \in \Omega \doteq \{\gamma_9(t) < x < 0, 0 < t < t_D\} \cup \{\gamma_{10}(t) < x < 0, t_D \leq t < t_Q\}$$

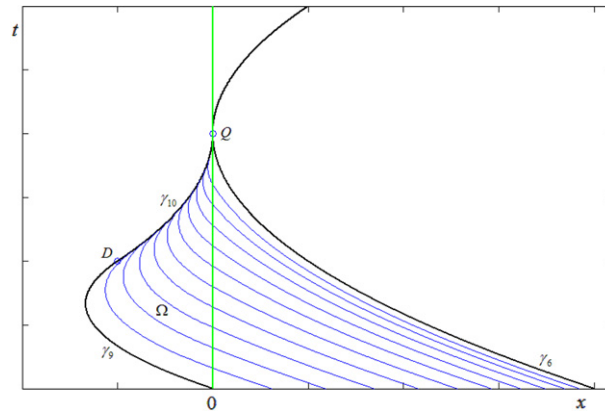


Fig. 2.3. In case $\frac{g^+}{g^-} \in (0, \frac{1}{2}]$, the characteristic curves issuing from $\{0 < x < \frac{(u_0^+)^2}{2g^+}, t = 0\}$ move into the second quadrant and will not enter the first quadrant again.

where $t_D = \frac{u_0^+}{g^+ - g^-}$ and γ_9, γ_{10} are given by

$$\gamma_9(t) = \frac{1}{2}g^-t^2 + u_0^+t, \quad \gamma_{10}(t) = \frac{(g^+t + u_0^+)^2}{2(2g^+ - g^-)}. \quad (2.19)$$

If $g^- = 2g^+$, then $\Omega \doteq \{\gamma_9(t) < x < 0, 0 < t < t_Q\}$. These are shown in Fig. 2.3 where $\gamma_6(t) = \frac{1}{2}g^+(t + \frac{u_0^+}{g^+})^2$.

Because the region Ω is below the line $\{t = t_Q\}$, the shock front $x_1(t)$ will get into the right side of t -axis from a point below Q . Therefore, by a direct calculation we obtain the solution (2.15). \square

As the ratio $\frac{g^+}{g^-}$ increases over a critical number, a new shock may appear in the solution of (1.1) from the point $Q = (0, t_Q)$. This is given in next result, in which we define

$$u_7(x, t) = g^+t + \frac{g^-u_0^+}{2g^+ - g^-} + \frac{2(g^- - g^+)}{(2g^+ - g^-)(2g^+ - 3g^-)} \left(2g^+(g^+ - g^-)t + g^-u_0^+ \right. \\ \left. - \sqrt{(g^+g^-t + g^-u_0^+)^2 + 2g^+(2g^+ - 3g^-)(2g^+ - g^-)x} \right) \quad (2.20)$$

for simplicity of notations.

Theorem 2.4. Suppose that $0 > u_0^- > u_0^+, \frac{u_0^+}{u_0^- + u_0^+} < \frac{g^+}{g^-} < 1$ then the problem (1.1) has a piecewise \mathcal{C}^1 solution:

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^3 \ (i = 1, 2, 3, 4, 7) \quad (2.21)$$

where u_i ($i = 1, 2, 3, 4$), u_7 are given in (2.2), (2.2), (2.1), (2.14), (2.20) respectively and the regions R_i ($i = 1, 2, 3, 4, 7$) are

$$R_1^3 = \{x < x_1(t), 0 \leq t < t_P\} \cup \{x < 0, t \geq t_P\} \\ R_2^3 = \{x > 0, 0 \leq t < t_Q\} \cup \{x > x_3(t), t_Q \leq t < t_A\} \cup \{x > x_4(t), t \geq t_A\} \\ R_3^3 = \{0 < x < x_2(t), t_P \leq t < t_A\} \cup \{0 < x < x_4(t), t \geq t_A\} \\ R_4^3 = \{x_1(t) < x < 0, 0 \leq t < t_P\} \\ R_7^3 = \{0 < x < x_3(t), t_Q \leq t < t_P\} \cup \{x_2(t) < x < x_3(t), t_P \leq t < t_A\}$$

where $t_Q = -\frac{u_0^+}{g^+}$, the shock front $x = x_1(t)$ is given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)) & t > 0 \\ x_1(0) = 0, \end{cases} \quad (2.22)$$

the point $P = (0, t_P)$ with $t_P > 0$ is the intersection point of the front $x = x_1(t)$ with the t -axis, the shock fronts $x = x_2(t)$ and $x = x_3(t)$ are determined by

$$\begin{cases} x_2(t) = \frac{1}{2}(u_3(x_2(t), t) + u_7(x_2(t), t)) & t > t_P \\ x_2(t_P) = 0 \end{cases} \quad (2.23)$$

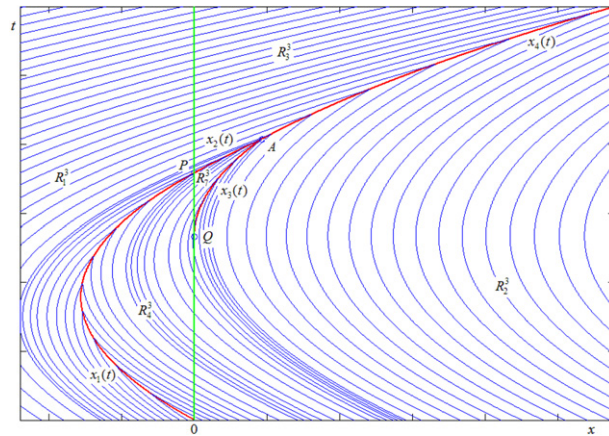


Fig. 2.4. A new shock front $x = x_3(t)$ forms from the point Q , and merges with the main front $x = x_2(t)$ into $x = x_4(t)$.

and

$$\begin{cases} x_3(t) = \frac{1}{2}(u_7(x_3(t), t) + u_2(x_3(t), t)) & t > t_Q \\ x_3(t_Q) = 0 \end{cases} \quad (2.24)$$

respectively, with $A = (x_A, t_A)$ being their intersection point, and the front $x = x_4(t)$ is given by

$$\begin{cases} x_4(t) = \frac{1}{2}(u_3(x_4(t), t) + u_2(x_4(t), t)) & t > t_A \\ x_4(t_A) = x_A. \end{cases} \quad (2.25)$$

The regions R_i^3 ($i = 1, 2, 3, 4, 7$), fronts $x = x_i(t)$ $i = 1, 2, 3, 4$ and the characteristics of the solution (2.21) are shown in Fig. 2.4.

Proof. From the Rankine–Hugoniot jump condition, we have

$$x'_1(0) = \frac{1}{2}(u_0^- + u_0^+) < 0 \quad (2.26)$$

so $x'_1(t) < 0$ for small t , which implies that the characteristic curves issuing from $\{x > 0, t = 0\}$ enter the left part of t -axis in a neighborhood of the origin.

By the method used in the proof of Theorem 2.3, we know that the characteristic curves issuing from each point on the segment $[0, t_Q]$ of t -axis determine the value of u in the region $\{\gamma_9(t) < x < 0, 0 < t < t_Q\}$ where $\gamma_9(t)$ is given by (2.19), and

$$t_Q = -\frac{2u_0^+}{g^-}. \quad (2.27)$$

These are illustrated in Fig. 2.5.

The value of u on the segment $[t_Q, t_Q]$ of t -axis is $u = u_4(0, t)$. By using $\frac{1}{2} < \frac{u_0^+}{u_0^- + u_0^+} < \frac{g^+}{g^-} < 1$ and a direct computation, we can determine the characteristic curves issuing from each point of the segment $[t_Q, t_Q]$ of t -axis, and the value of u in the region:

$$\Omega_1 \doteq \{0 < x < \gamma_{12}(t), t_Q \leq t < t_Q\} \cup \{\gamma_{14}(t) < x < \gamma_{12}(t), t_Q \leq t \leq t_B\}$$

where $t_B = -\frac{4(g^+)^2 - 6g^+g^-(g^-)^2}{2g^+g^-(g^+ - g^-)}u_0^+$, $\gamma_{12}(t)$ and $\gamma_{14}(t)$ are given by

$$\gamma_{12}(t) = -\frac{(g^+g^-t + g^-u_0^+)^2}{2g^+(2g^+ - 3g^-)(2g^+ - g^-)}, \quad (2.28)$$

$$\gamma_{14}(t) = \frac{1}{2}g^+t^2 + \frac{2g^+ - g^-}{g^-}u_0^+t + \frac{2(g^+ - g^-)}{(g^-)^2}(u_0^+)^2. \quad (2.29)$$

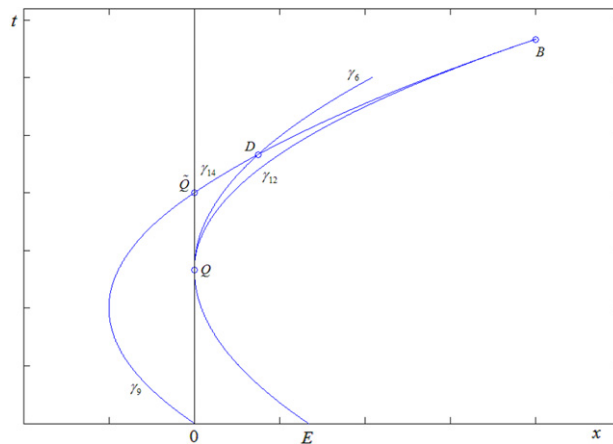


Fig. 2.5. The overlapping area of the characteristic curves between γ_6 and γ_{12} implies that a new shock front in this region will appear from the point Q .

On the other hand, the characteristic curves issuing from $\{x > 0, t = 0\}$ can determine the value of u on Ω_2 ,

$$\Omega_2 \doteq \{x > 0, 0 \leq t < t_Q\} \cup \{x > \gamma_6(t), t \geq t_Q\}$$

with $\gamma_6(t)$ being given by (2.19). Noting that the $\Omega_1 \cap \Omega_2$ is a crooked triangle $\triangle QDP$ with D being the intersection point of $x = \gamma_{11}(t)$ and $x = \gamma_6(t)$, we know that a shock front $x = x_3(t)$ given by (2.24) will form from the point Q in the region $\Omega_1 \cap \Omega_2$, because the characteristic curves starting from $\{x = 0, t_Q < t < t_{\bar{Q}}\}$ and $\{t = 0, x > x_E\}$ intersect in this region with $x_E = \frac{(u_0^+)^2}{2g^+}$.

Now, we study the solution behavior determined by the initial data in $\{x < 0, t = 0\}$. From the Rankine–Hugoniot jump condition, the shock front $x = x_1(t)$ is given by (2.22). We first assert that the front $x = x_1(t)$ will enter the right part of the t -axis from a point P which is above the point Q , i.e. $x_1(t_Q) < 0$. In fact, $x_1'(t)$ can be written as

$$x_1'(t) = \frac{1}{2}(g^-t + u_0^+ + (g^+ - g^-)t_0 + g^-t + u_0^-)$$

where $t_0 \in [0, t_Q]$ and $g^+ - g^- < 0$, so

$$x_1'(t) \leq \frac{1}{2}(2g^-t + u_0^+ + u_0^-)$$

which implies

$$x_1(t) \leq \frac{1}{2}g^-t^2 + \frac{1}{2}(u_0^+ + u_0^-)t. \quad (2.30)$$

By assumption $\frac{u_0^+}{u_0^- + u_0^+} < \frac{g^+}{g^-}$, from (2.30) we have $x(t_Q) \leq \frac{1}{2}g^-(-\frac{u_0^+}{g^+})^2 + \frac{1}{2}(u_0^+ + u_0^-)(-\frac{u_0^+}{g^+}) < 0$.

After crossing into the right side of t -axis, the main shock front becomes $x = x_2(t)$ given by (2.23), then it intersects with $x = x_3(t)$ at the point A , and forms a new shock front $x = x_4(t)$ being given by (2.25). In this way, we obtain the explicit solution to the initial value problem. \square

As a special case, from Theorem 2.4 we have the following result, which shows that the new shock front $x = x_3(t)$ is mainly created by the discontinuous source term.

Theorem 2.5. Suppose that $u_0^- = u_0^+ < 0$, and $\frac{1}{2} < \frac{g^+}{g^-} < 1$, then the solution to the problem (1.1) is given explicitly as,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^4 \quad (i = 1, 2, 3, 4, 7) \quad (2.31)$$

where u_i ($i = 1, 2, 3, 4$), u_7 are defined in (2.2), (2.2), (2.1), (2.14), (2.20) respectively and the regions R_i^4 ($i = 1, 2, 3, 4, 7$) are given by

$$R_1^4 = \{x < \gamma_9(t), 0 \leq t < t_{\bar{Q}}\} \cup \{x < 0, t \geq t_{\bar{Q}}\}$$

$$R_2^4 = \{x > 0, 0 \leq t < t_Q\} \cup \{x > x_1(t), t_Q \leq t < t_A\} \cup \{x > x_2(t), t \geq t_A\}$$

$$R_3^4 = \{0 < x < \gamma_{14}(t), t_{\bar{Q}} \leq t < t_A\} \cup \{0 < x < x_2(t), t \geq t_A\}$$

$$R_4^4 = \{\gamma_9(t) < x < 0, 0 \leq t < t_{\bar{Q}}\}$$

$$R_7^4 = \{0 < x < x_1(t), t_Q \leq t < t_{\bar{Q}}\} \cup \{\gamma_{14}(t) < x < x_1(t), t_{\bar{Q}} \leq t < t_A\}$$

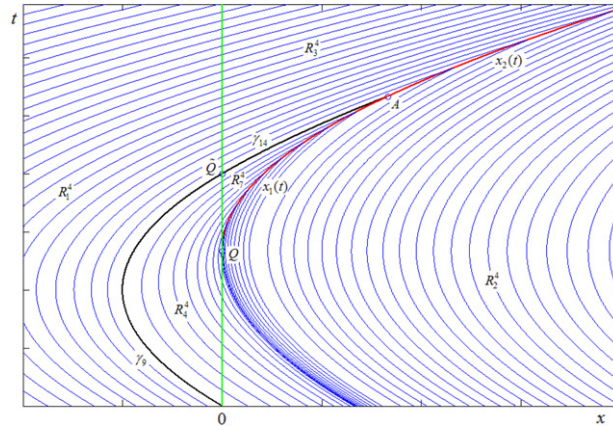


Fig. 2.6. There is only one shock front issuing from Q and $x = \gamma_9(t)$ is a weak discontinuity.

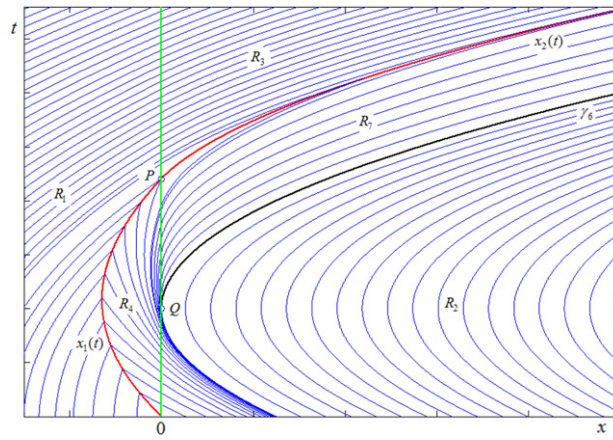


Fig. 2.7. When $\frac{g^+}{g^-} \geq 1$, the new shock in solution (2.21) disappears and a weak discontinuity $x = \gamma_6(t)$ appears.

where t_Q and t_Q^- are the same as in the proof of [Theorem 2.4](#), $\gamma_9(t)$ and $\gamma_{14}(t)$ are defined in (2.19) and (2.29) respectively, $x_1(t)$ and $x_2(t)$ are given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_7(x_1(t), t) + u_2(x_1(t), t)) & t > t_Q \\ x_1(t_Q) = 0 \end{cases}$$

and

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_7(x_2(t), t) + u_2(x_2(t), t)) & t > t_A \\ x_2(t_A) = x_A \end{cases}$$

with $A = (x_A, t_A)$ is the intersection of $x = x_1(t)$ and $x = \gamma_{14}(t)$. This solution has weak singularities on $\{x = \gamma_9(t)\}$, $\{x = \gamma_{14}(t)\}$ and $\{x = 0\}$, and $\{x = x_1(t)\}$ and $\{x = x_2(t)\}$ are two shock fronts. The regions R_i^4 ($i = 1, 2, 3, 4, 7$), fronts $x_i(t)$ ($i = 1, 2$) and the characteristics of the solution (2.31) are shown in [Fig. 2.6](#).

Remark 2.2. In [Theorem 2.4](#), if $\frac{g^+}{g^-} \geq 1$, then after computation we have

$$\Omega_1 \doteq \{0 < x < \gamma_6(t), t_Q \leq t < t_Q^-\} \cup \{\gamma_{14}(t) < x < \gamma_6(t), t \geq t_Q^-\}.$$

Therefore, Ω_1 will not intersect with Ω_2 and the shock front $x_3(t)$ will disappear. The local solution of (1.1) in this condition is show in [Fig. 2.7](#). We still have $u = u_i(x, t)$ as $x \in R_i$ ($i = 1, 2, 3, 4, 7$) but the curve γ_6 becomes a weak discontinuity. We should note that when t goes to infinity γ_6 may interact with the main shock front $x_2(t)$ and then disappear, in which the region R_7 is bounded.

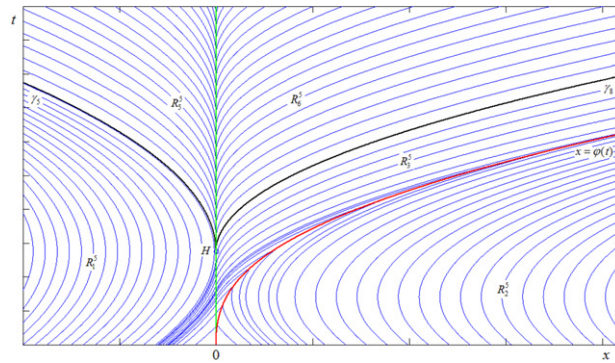


Fig. 2.8. R_5^5 and R_6^5 are “vacuum” regions.

2.2. Propagation of a shock with vacuum and a possible bifurcation

In this subsection, we construct a solution to (1.1), which has a “vacuum” region as we explained in Section 1, when $g^- < 0 < g^+$ and t is sufficiently large.

In this subsection, we shall always assume

$$g^- < 0 < g^+, \quad u_0^- > 0 > u_0^+. \quad (2.32)$$

When u_0^- decreases from $-u_0^+$ to $\frac{g^-}{g^+}u_0^+$, the shock front in the solution of (1.1) will change from being bent to the right to being bent to the left. Let the point P be the intersection of the shock front and the positive t -axis, $Q = (0, -\frac{u_0^+}{g^+})$ be given in Theorem 2.4, and denote by H the point

$$H = (0, t_H) = \left(0, -\frac{u_0^-}{g^-}\right). \quad (2.33)$$

We shall see that the pattern of the solution to (1.1) depends on the relative positions of the points P , H and Q . This is described from Theorems 2.6–2.10. For convenience of presentation, we define

$$u_5(x, t) = -\sqrt{2g^-x} \quad u_6(x, t) = \sqrt{2g^+x} \quad (2.34)$$

$$\gamma_5(t) = \frac{1}{2}g^- \left(t + \frac{u_0^-}{g^-}\right)^2 \quad \gamma_8(t) = \frac{1}{2}g^+ \left(t + \frac{u_0^+}{g^+}\right)^2. \quad (2.35)$$

First, let us consider the case $u_0^- + u_0^+ \geq 0$, in which the shock enters the first quadrant from the origin and a “vacuum” region will appear from the point H and we determine the solution of (4.24) in the “vacuum” region by moving the characteristic curves $\gamma_5(t)$ and $\gamma_8(t)$ along the t -axis direction to fill the region $R_5^5 \cup R_6^5$ (see Fig. 2.8), in this way we obtain a global entropy solution to the original problem (1.1), and we shall show the uniqueness and stability of this entropy solution at the end of this work.

Theorem 2.6. Under the assumption (2.32) and $u_0^- + u_0^+ \geq 0$, the problem (1.1) has a piecewise C^1 solution,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^5 \quad (i = 1, 2, 3, 5, 6) \quad (2.36)$$

where u_1, u_2, u_3, u_5, u_6 are given by (2.2), (2.2), (2.1), (2.34), (2.34) respectively and the regions R_i^5 ($i = 1, 2, 3, 5, 6$) are defined as

$$\begin{aligned} R_1^5 &= \{x < 0, 0 \leq t < t_H\} \cup \{x < \gamma_5(t), t \geq t_H\} \\ R_2^5 &= \{x > \varphi(t), t \geq 0\} \\ R_3^5 &= \{0 < x < \varphi(t), 0 \leq t < t_H\} \cup \{\gamma_8(t) < x < \varphi(t), t \geq t_H\} \\ R_5^5 &= \{\gamma_5(t) < x < 0, t > t_H\} \\ R_6^5 &= \{0 < x < \gamma_8(t), t > t_H\}. \end{aligned}$$

The regions R_i^5 ($i = 1, 2, 3, 5, 6$), the curve $\varphi(t)$ and the characteristics of the solution (2.36) are shown in Fig. 2.8.

Proof. By the method of characteristics we can obtain the solution u in R_1^5 and R_2^5 immediately. By the assumption that $u_0^- + u_0^+ \geq 0$, the shock front $x = \varphi(t)$ satisfies

$$\varphi'(0) = \frac{1}{2}(u_0^- + u_0^+) \geq 0.$$

Because $g^+ + g^- > 0$, we know $u_1(x, t) + u_2(x, t) > 0$ for $t > 0$ which implies that the shock front will enter the right part of t -axis. By calculation we get $u = u_3(x, t)$ in R_3^5 .

Noting that the characteristic curves issuing from the x -axis cannot reach the region $R_5^5 \cup R_6^5$ ("vacuum"), we can move the characteristic curves $\gamma_5(t)$ and $\gamma_8(t)$ in an upward direction to determine the characteristic curves in these two regions, that is, the characteristic curves in R_5^5 and R_6^5 are

$$x(t; t_0) = \frac{1}{2}g^-(t + t_0)^2 \quad (2.37)$$

$$x(t; t_0) = \frac{1}{2}g^+(t + t_0)^2 \quad (2.38)$$

when $t_0 > t_H$. Therefore, from $u(x(t), t) = x'(t)$ we obtain $u = u_5(x, t)$ and $u = u_6(x, t)$ in R_5^5 and R_6^5 respectively. In this way, we obtain a global entropy solution to the initial value problem. \square

Now, if u_0^- in Theorem 2.6 decreases to satisfy $u_0^- + u_0^+ < 0$, the shock front will first enter the second quadrant from the origin. We assume this shock front returns back to the first quadrant at the P on t -axis, the following theorem studies the case in which P is below Q .

Theorem 2.7. Under the assumption that (2.32), $u_0^- + u_0^+ < 0$ and $\frac{u_0^-}{g^-} < \frac{u_0^+}{g^+}$ hold. The shock front $x = x_1(t)$ is given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)) & t > 0 \\ x_1(0) = 0. \end{cases}$$

If there exists $t_p > 0$, satisfying $x_1(t_p) = 0$ and $t_p < t_Q$ (see Fig. 2.9 where $\gamma_6(t)$ is given by (2.19)) then the problem (1.1) has a global entropy solution,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^6 \quad (i = 1, 2, 3, 4, 5, 6) \quad (2.39)$$

where u_i , ($1 \leq i \leq 6$) are given by (2.2), (2.2), (2.1), (2.14), (2.34), (2.34) respectively, and the regions R_i^6 ($i = 1, 2, 3, 4, 5, 6$) are defined as

$$\begin{aligned} R_1^6 &= \{x < x_1(t), 0 \leq t < t_p\} \cup \{x < 0, t_p \leq t < t_H\} \cup \{x < \gamma_5(t), t \geq t_H\} \\ R_2^6 &= \{x > 0, 0 \leq t < t_p\} \cup \{x > x_2(t), t \geq t_p\} \\ R_3^6 &= \{0 < x < x_2(t), t_p \leq t < t_H\} \cup \{\gamma_8(t) < x < x_2(t), t \geq t_H\} \\ R_4^6 &= \{x_1(t) < x < 0, 0 \leq t < t_p\} \\ R_5^6 &= \{\gamma_5(t) < x < 0, t \geq t_H\} \\ R_6^6 &= \{0 < x < \gamma_8(t), t \geq t_H\} \end{aligned}$$

where t_H , $\gamma_5(t)$ and $\gamma_8(t)$ are given by (2.33), (2.35) and (2.35) respectively, and the shock front $x = x_2(t)$ is defined as

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_3(x_2(t), t) + u_2(x_2(t), t)) & t > t_p \\ x_2(t_p) = 0. \end{cases}$$

The regions R_i^6 ($1 \leq i \leq 6$), the curves $x = x_1(t)$, $x = x_2(t)$ and the characteristics of the solution (2.39) are shown in Fig. 2.10.

This theorem can be obtained in the same way as that given in Theorem 2.6 by integration along characteristics.

Denote by $x = \gamma_7(t)$ with

$$\gamma_7(t) = \frac{1}{2}g^-\left(t + \frac{u_0^+}{g^+}\right)^2 \quad (2.40)$$

the characteristic issuing from the point $Q = (0, -\frac{u_0^+}{g^+})$.

If u_0^- in Theorem 2.7 continues to decrease, the shock front $x = x_1(t)$ will intersect with $x = \gamma_7(t)$ (see Fig. 2.11), and a "vacuum" region will appear from the point Q . Denote by u^* the critical value of u_0^- , such that when $u_0^- = u^*$, the point

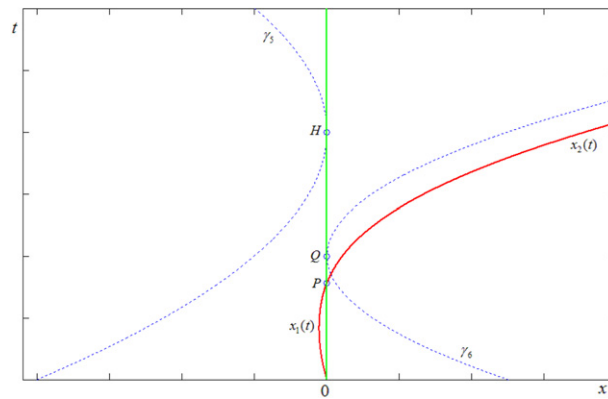


Fig. 2.9. The shock front moves into the first quadrant from the point P which is below the point Q .

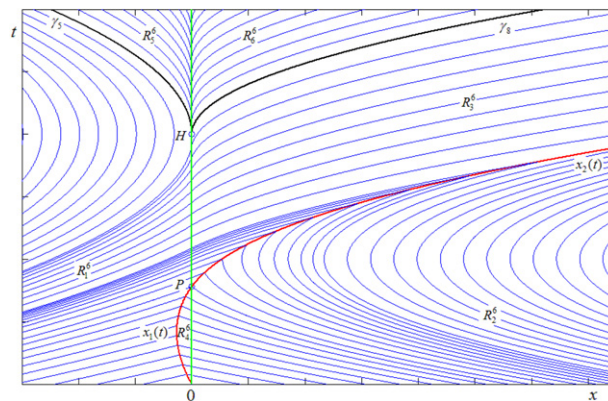


Fig. 2.10. R_5^6 and R_6^6 are "vacuum" regions.

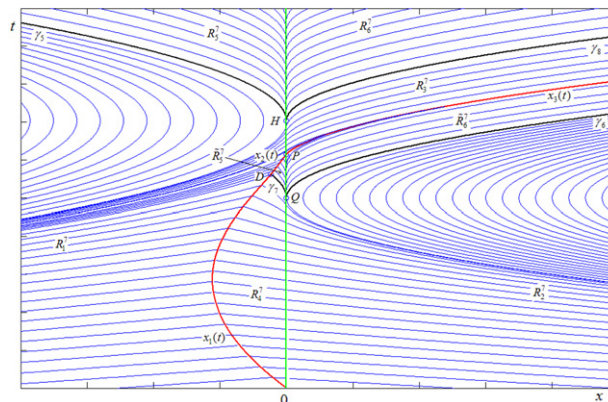


Fig. 2.11. \tilde{R}_5^7 , \tilde{R}_6^7 , R_5^7 and R_6^7 are "vacuum" regions.

P coincides with Q (see Fig. 2.9). So, when $u_0^- < u^*$ and $u^* - u_0^-$ is small enough, we know that the shock front $x = x_1(t)$ will move into the first quadrant from some point between Q and H and then another "vacuum" region will appear from H . These are shown in the following theorem.

Theorem 2.8. Suppose that $u_0^- + u_0^+ < 0$, $\frac{u_0^-}{g^-} < \frac{u_0^+}{g^+}$ and the assumption (2.32) hold. The shock front $x = x_1(t)$ is given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)) & t > 0 \\ x_1(0) = 0. \end{cases}$$

Let D be the intersection point of $x = x_1(t)$ and $x = \gamma_7(t)$, with $D = (x_D, t_D)$ satisfying $x_D < 0$ and $t_D > t_Q$. The shock front $x = x_2(t)$ is given by

$$\begin{cases} x'_2(t) = \frac{1}{2}(u_1(x_2(t), t) + u_5(x_2(t), t)) & t > t_D \\ x_2(t_D) = x_D. \end{cases}$$

If there exists $t_p \in (t_Q, t_H)$ where t_H is given by (2.33), satisfying $x_2(t_p) = 0$ and $t_p > \frac{u_0^+ - u_0^-}{g^- - g^+}$ then the problem (1.1) has a global entropy solution given by,

$$u(x, t) = \begin{cases} u_i(x, t), & \text{as } (x, t) \in R_i^7 \ (i = 1, 2, 3, 4, 5, 6) \\ u_i(x, t), & \text{as } (x, t) \in \tilde{R}_i^7 \ (i = 5, 6) \end{cases} \quad (2.41)$$

where u_i ($1 \leq i \leq 6$) are given by (2.2), (2.2), (2.1), (2.14), (2.34), (2.34) respectively and the regions R_i^7 ($i = 1, \dots, 6$) and \tilde{R}_i^7 ($i = 5, 6$) are defined as

$$\begin{aligned} R_1^7 &= \{x < \varphi(t), 0 \leq t < t_p\} \cup \{x < 0, t_p \leq t < t_H\} \cup \{x < \gamma_5(t), t \geq t_H\} \\ R_2^7 &= \{x > 0, 0 \leq t < t_Q\} \cup \{x > \gamma_6(t), t \geq t_Q\} \\ R_3^7 &= \{0 < x < \varphi(t), t_p \leq t < t_H\} \cup \{\gamma_8(t) < x < \varphi(t), t \geq t_H\} \\ R_4^7 &= \{\varphi(t) < x < 0, 0 \leq t < t_Q\} \cup \{\varphi(t) < x < \gamma_7(t), t_Q \leq t < t_D\} \\ R_5^7 &= \{\gamma_5(t) < x < 0, t \geq t_H\} \\ R_6^7 &= \{0 < x < \gamma_8(t), t \geq t_H\} \\ \tilde{R}_5^7 &= \{\gamma_7(t) < x < 0, t_Q \leq t < t_D\} \cup \{\varphi(t) < x < 0, t_D \leq t < t_p\} \\ \tilde{R}_6^7 &= \{0 < x < \gamma_6(t), t_Q \leq t < t_p\} \cup \{\varphi(t) < x < \gamma_6(t), t \geq t_p\} \end{aligned}$$

where $\gamma_5(t)$, $\gamma_6(t)$, $\gamma_7(t)$ and $\gamma_8(t)$ are given by (2.35), (2.19), (2.40), (2.35) respectively and the shock front $x = \varphi(t)$ is given by

$$\varphi(t) = \begin{cases} x_1(t) & 0 \leq t < t_D \\ x_2(t) & t_D \leq t < t_p \\ x_3(t) & t \geq t_p, \end{cases}$$

with $x_3(t)$ being determined by

$$\begin{cases} x'_3(t) = \frac{1}{2}(u_3(x_3(t), t) + u_6(x_3(t), t)) & t > t_p \\ x_3(t_p) = x_p. \end{cases}$$

The regions R_i^7 ($i = 1, \dots, 6$), \tilde{R}_i^7 ($i = 5, 6$), the curves $x_i(t)$, ($i = 1, 2, 3, 4$) and the characteristics of the solution (2.41) are shown in Fig. 2.11.

As before, this result can be obtained in the same way as that given in Theorem 2.6 by integration along characteristics.

Remark 2.3. From $-\frac{u_0^-}{g^-} > -\frac{u_0^+}{g^+}$, we have $t_H > \frac{u_0^+ - u_0^-}{g^- - g^+}$, so it is possible for t_p to satisfy $t_p > \frac{u_0^+ - u_0^-}{g^- - g^+}$. Moreover, under this condition, we can prove that $x = x_3(t)$ will not intersect with $x = \gamma_6(t)$ on $[t_p, +\infty)$. In fact, by a simple computation we have

$$x'_3(t) = \frac{1}{2}(g^+t + u_0^- + (g^- - g^+)t_{01} + g^+t - g^+t_{02})$$

where $t_{01} \in (t_p, t_H)$ and $t_{02} \in (t_Q, t_p)$. Define $F(t) = x_3(t) - \gamma_6(t)$ and then we have $F(t_p) < 0$ and

$$\begin{aligned} F'(t) &= \frac{1}{2}(u_0^- + (g^- - g^+)t_{01} - g^+t_{02} - 2u_0^+) \\ &\leq \frac{1}{2}\left(u_0^- + (g^- - g^+)t_p - g^+\left(-\frac{u_0^+}{g^+}\right) - 2u_0^+\right) \\ &= \frac{1}{2}(u_0^- + (g^- - g^+)t_p - u_0^+). \end{aligned}$$

From the condition $t_p > \frac{u_0^+ - u_0^-}{g^- - g^+}$, for all $t \geq t_p$ we have $F'(t) < 0$. Therefore, $F(t) < 0$ for all $t \geq t_p$, i.e. $x = x_3(t)$ will never intersect with $x = \gamma_6(t)$ on $[t_p, +\infty)$.

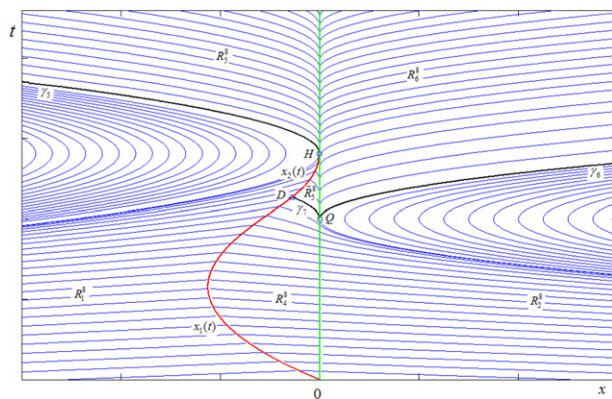


Fig. 2.12. The shock front $x = x_2(t)$ disappears at the point H .

Remark 2.4. In Theorem 2.7, if $t_P \in (t_Q, t_H)$ and $t_P - t_Q$ is small enough, by numerical simulation it shows that the shock front $x = x_3(t)$ may intersect with the curve $x = \gamma_6(t)$ when t is sufficiently large.

If u_0^- in Theorem 2.8 continues to decrease, the point P will coincide with H , this is a critical case of the solution to (1.1), in which the shock front disappears at the point H (see Fig. 2.12). This critical case is unstable usually, as a small change of u_0^- or other parameters can cause the shock front turn to the left or right from some point in the neighborhood of H . These are shown in the following theorem.

Theorem 2.9. Suppose (2.32), and $u_0^- + u_0^+ < 0$, $\frac{u_0^-}{g^-} < \frac{u_0^+}{g^+}$ hold. The shock front $x = x_1(t)$ is given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)) & t > 0 \\ x_1(0) = 0. \end{cases}$$

Let $D = (x_D, t_D)$ be the intersection point of $x = x_1(t)$ and $x = \gamma_7(t)$, with $x_D < 0$ and $t_D > t_Q$. The shock front $x = x_2(t)$ is defined as

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_1(x_2(t), t) + u_5(x_2(t), t)) & t > t_D \\ x_2(t_D) = x_D. \end{cases}$$

If $x_2(t_H) = 0$ with t_H being given by (2.33), then the problem (1.1) has a global entropy solution given by,

$$u(x, t) = \begin{cases} u_i(x, t), & \text{as } (x, t) \in R_i^8 \ (i = 1, 2, 4, 5, 6) \\ u_5(x, t), & \text{as } (x, t) \in \tilde{R}_5^8 \end{cases} \quad (2.42)$$

where u_1, u_2, u_4, u_5, u_6 are given by (2.2), (2.2), (2.14), (2.34), (2.34) respectively and the regions R_i^8 ($i = 1, 2, 4, 5, 6$) and \tilde{R}_5^8 are defined as

$$\begin{aligned} R_1^8 &= \{x < x_1(t), 0 \leq t < t_D\} \cup \{x < x_2(t), t_D \leq t < t_H\} \cup \{x < \gamma_5(t), t \geq t_H\} \\ R_2^8 &= \{x > 0, 0 \leq t < t_Q\} \cup \{x > \gamma_6(t), t \geq t_Q\} \\ R_4^8 &= \{x_1(t) < x < 0, 0 \leq t < t_Q\} \cup \{x_1(t) < x < \gamma_7(t), t_Q \leq t < t_D\} \\ R_5^8 &= \{\gamma_5(t) < x < 0, t \geq t_H\} \\ R_6^8 &= \{0 < x < \gamma_6(t), t \geq t_Q\} \\ \tilde{R}_5^8 &= \{\gamma_7(t) < x < 0, t_Q \leq t < t_D\} \cup \{x_2(t) < x < 0, t_D \leq t < t_H\} \end{aligned}$$

where γ_6, γ_7 are given by (2.19) and (2.40) respectively. The regions R_i^8 ($i = 1, 2, 4, 5, 6$), \tilde{R}_5^8 , the curves $x_i(t)$, ($i = 1, 2$) and the characteristics of the solution (2.42) are shown in Fig. 2.12.

If u_0^- in Theorem 2.9 keeps to decrease, the shock front will not enter the first quadrant and the “vacuum” region appears from the point Q (see Fig. 2.13). These are shown in the following theorem.

Theorem 2.10. Suppose that $u_0^- + u_0^+ < 0$, $\frac{u_0^-}{g^-} < \frac{u_0^+}{g^+}$ and (2.32) hold. The shock front $x = x_1(t)$ is given by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_1(x_1(t), t) + u_4(x_1(t), t)) & t > 0 \\ x_1(0) = 0. \end{cases}$$

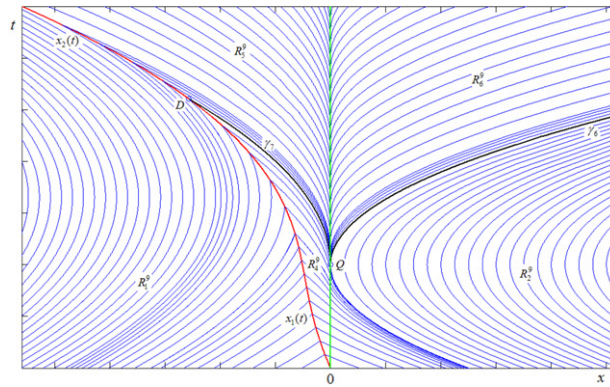


Fig. 2.13. The shock front does not move into the first quadrant.

The intersection point of $x = x_1(t)$ and $x = \gamma_7(t)$ is $D = (x_D, t_D)$ with $x_D < 0$ and $t_D > t_Q$. The shock front $x = x_2(t)$ is defined as

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_1(x_2(t), t) + u_5(x_2(t), t)) & t > t_D \\ x_2(t_D) = x_D. \end{cases}$$

If $x_2(t) < 0$ for all $t \geq t_D$, then the problem (1.1) has a global entropy solution as follows,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^9 \quad (i = 1, 2, 4, 5, 6) \quad (2.43)$$

where u_1, u_2, u_4, u_5, u_6 are given by (2.2), (2.2), (2.14), (2.34), (2.34) respectively and the regions R_i^9 ($i = 1, 2, 4, 5, 6$) are defined as

$$\begin{aligned} R_1^9 &= \{x < x_1(t), 0 \leq t < t_D\} \cup \{x < x_2(t), t \geq t_D\} \\ R_2^9 &= \{x > 0, 0 \leq t < t_Q\} \cup \{x > \gamma_6(t), t > t_Q\} \\ R_4^9 &= \{x_1(t) < x < 0, 0 \leq t < t_Q\} \cup \{x_1(t) < x < \gamma_7(t), t_Q \leq t < t_D\} \\ R_5^9 &= \{\gamma_7(t) < x < 0, t_Q \leq t < t_D\} \cup \{x_2(t) < x < 0, t \geq t_D\} \\ R_6^9 &= \{0 < x < \gamma_6(t), t > t_Q\} \end{aligned}$$

where $\gamma_6(t)$ and $\gamma_7(t)$ are given by (2.19) and (2.40) respectively. The regions R_i^9 ($i = 1, 2, 4, 5, 6$), the curves $x_i(t)$, ($i = 1, 2$) and the characteristics of solution (2.43) are shown in Fig. 2.13.

As before, these two results can be obtained in the same way as that given in Theorem 2.6 by integration along characteristics.

3. The case of the initial data generating a rarefaction wave

In this section, we are going to study the influence of the discontinuous source term on the rarefaction waves. There are three phenomena. The first one is that the source term just bends the rarefaction wave. The second one is that the source term creates a “vacuum” region. The third one is that the source term creates a new shock. We always assume in this section that $g^- + g^+ > 0$ and $u_0^- < u_0^+$.

3.1. Bend of a rarefaction wave without shock

In this subsection, we assume $g^- < g^+$, under which there will be no shock in the solution to (1.1).

When $g^+ > g^- > 0$, the source term just bends the rarefaction wave, and yields some weak discontinuities in states, i.e. the problem (1.1) has a continuous solution without “vacuum” region. These will be described from Theorems 3.1–3.3. For convenience, we introduce

$$u_9(x, t) = \frac{1}{2}g^+t + \frac{x}{t} \quad (3.1)$$

$$\gamma_2(t) = \frac{1}{2}g^+t^2 + u_0^+t \quad \gamma_{13}(t) = \frac{1}{2}g^+t^2 + u_0^-t. \quad (3.2)$$

The following theorem considers the case that the rarefaction wave is in the right part of the discontinuity of the source term.

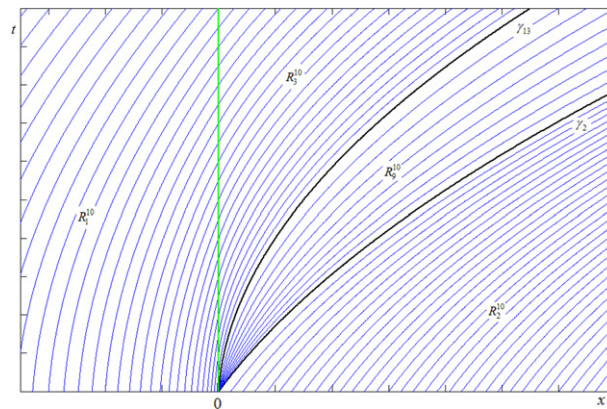


Fig. 3.1. The rarefaction wave R_9^{10} with a weak discontinuity on $\{x = 0\}$.

Theorem 3.1. Suppose $g^+ > g^- > 0$ and $0 < u_0^- < u_0^+$, then the problem (1.1) has a global curved rarefaction wave with a weak discontinuity on $\{x = 0\}$,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{10} \quad (i = 1, 2, 3, 9) \quad (3.3)$$

where u_1, u_2, u_3, u_9 are given by (2.2), (2.2), (2.1), (3.1) respectively and the regions R_i^{10} ($i = 1, 2, 3, 9$) are defined as

$$\begin{aligned} R_1^{10} &= \{x < 0, t \geq 0\} & R_2^{10} &= \{x > \gamma_2(t), t \geq 0\} \\ R_3^{10} &= \{0 < x < \gamma_{13}(t), t \geq 0\} & R_9^{10} &= \{\gamma_{13}(t) < x < \gamma_2(t), t \geq 0\} \end{aligned}$$

where $\gamma_2(t)$ and $\gamma_{13}(t)$ are given in (3.2). The regions R_i^{10} ($i = 1, 2, 3, 9$) and the characteristics of the solution (3.3) are shown in Fig. 3.1.

Proof. By integration along characteristics, we can immediately deduce u_1, u_2 and u_3 in their corresponding regions. In order to determine the value of u in R_9^{10} , let us consider a characteristic curve $x = x(t)$ generating from the origin with the slope $c \in (u_0^-, u_0^+)$ at the origin,

$$\begin{cases} x'(t) = u(x(t), t) \\ x(0) = 0 \\ x(t) > 0, \quad x'(0) = c \end{cases}$$

which implies $x(t) = \frac{1}{2}g^+t^2 + ct$ and $u(x, t) = \frac{1}{2}g^+t + \frac{x}{t}$. Thus, we get $u = u_9(x, t)$ in R_9^{10} . \square

Next, we are going to study the case that the fan of rarefaction wave near the origin containing the discontinuity of the source term (see Fig. 3.2).

For convenience of presentation, we denote by \tilde{H} the point

$$H = \left(0, -\frac{u_0^+}{g^+}\right) \quad (3.4)$$

and define

$$\gamma_1(t) = \frac{1}{2}g^-t^2 + u_0^-t \quad (3.5)$$

$$\gamma_{11}(t) = \frac{1}{2}g^+t^2 + \frac{2g^+ - g^-}{g^-}u_0^-t + \frac{2(g^+ - g^-)}{(g^-)^2}(u_0^-)^2 \quad (3.6)$$

$$u_8(x, t) = \frac{1}{2}g^-t + \frac{x}{t} \quad (3.7)$$

$$u_{10}(x, t) = g^+t + \frac{g^- - 2g^+}{2g^+ - 2g^-} \left(g^+ - \frac{1}{2}g^-t - \sqrt{\left(\frac{1}{2}g^-t\right)^2 + 2x(g^+ - g^-)} \right) \quad (3.8)$$

$$u_{11}(x, t) = \begin{cases} u_{10}(x, t), & x < \frac{1}{2}g^+t^2 \\ u_9(x, t), & x > \frac{1}{2}g^+t^2 \end{cases} \quad (3.9)$$

where u_9 is given in (3.1).

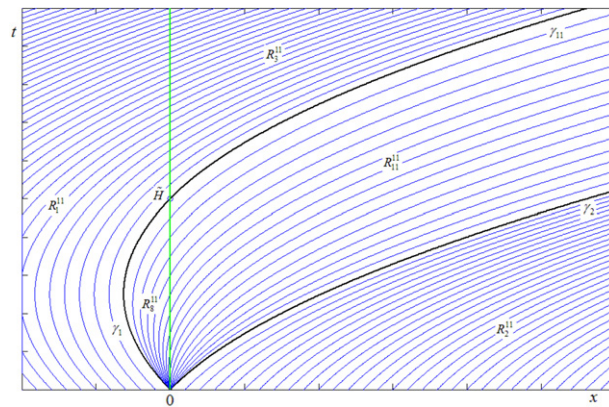


Fig. 3.2. The rarefaction wave $R_8^{11} \cup R_{11}^{11}$ with a weak discontinuity $\{x = 0\}$.

Theorem 3.2. Suppose $g^+ > g^- > 0$ and $u_0^- < 0 < u_0^+$, then the problem (1.1) a global curved rarefaction wave with a weak discontinuity on $\{x = 0\}$,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{11} \quad (i = 1, 2, 3, 8, 11) \quad (3.10)$$

where $u_1, u_2, u_3, u_8, u_{11}$ are given by (2.2), (2.2), (2.1), (3.7), (3.9) respectively and the regions R_i^{11} ($i = 1, 2, 3, 8, 11$) are defined as

$$\begin{aligned} R_1^{11} &= \{x < \gamma_1(t), 0 \leq t < t_{\bar{H}}\} \\ R_2^{11} &= \{x > \gamma_2(t), t \geq 0\} \\ R_3^{11} &= \{0 < x < \gamma_{11}(t), t \geq t_{\bar{H}}\} \\ R_8^{11} &= \{\gamma_1(t) < x < 0, 0 < t < t_{\bar{H}}\} \\ R_{11}^{11} &= \{0 < x < \gamma_2(t), 0 < t < t_{\bar{H}}\} \cup \{\gamma_{11}(t) < x < \gamma_2(t), t \geq t_{\bar{H}}\} \end{aligned}$$

where $t_{\bar{H}}, \gamma_1(t), \gamma_2(t), \gamma_{11}(t)$ are given in (3.4), (3.5), (3.2) and (3.6) respectively. The regions R_i^{11} ($i = 1, 2, 3, 8, 11$) and the characteristics of the solution (3.10) are shown in Fig. 3.2.

In the following theorem, we study the case that the rarefaction wave first locates in the left part of the discontinuity of the source term in a neighborhood of the origin and then move into the first quadrant (see Fig. 3.3). Notably, in the solution (3.11), γ_6 is a weak discontinuity.

Theorem 3.3. Suppose $g^+ > g^- > 0$ and $u_0^- < u_0^+ < 0$, then the problem (1.1) has a global curved rarefaction wave with two weak discontinuities:

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{12} \quad (i = 1, 2, 3, 4, 7, 8, 10) \quad (3.11)$$

where $u_1, u_2, u_3, u_4, u_7, u_8, u_{10}$ are given by (2.2), (2.2), (2.1), (2.14), (2.20), (3.7), (3.8) respectively and the regions R_i^{12} ($i = 1, 2, 3, 4, 7, 8, 10$) are defined as

$$\begin{aligned} R_1^{12} &= \{x < \gamma_1(t), 0 \leq t < t_{\bar{H}}\} \cup \{x < 0, t \geq t_{\bar{H}}\} \\ R_2^{12} &= \{x > 0, 0 \leq t < t_{\bar{Q}}\} \cup \{x > \gamma_6(t), t \geq t_{\bar{Q}}\} \\ R_3^{12} &= \{0 < x < \gamma_{11}(t), t \geq t_{\bar{H}}\} \\ R_4^{12} &= \{\gamma_9(t) < x < 0, 0 \leq t < t_{\bar{Q}}\} \\ R_7^{12} &= \{0 < x < \gamma_6(t), t_{\bar{Q}} \leq t < t_{\bar{Q}}\} \cup \{\gamma_{14}(t) < x < \gamma_6(t), t \geq t_{\bar{Q}}\} \\ R_8^{12} &= \{\gamma_1(t) < x < \gamma_9(t), 0 < t < t_{\bar{Q}}\} \cup \{\gamma_1(t) < x < 0, t_{\bar{Q}} \leq t < t_{\bar{H}}\} \\ R_{10}^{12} &= \{0 < x < \gamma_{14}(t), t_{\bar{Q}} \leq t < t_{\bar{H}}\} \cup \{\gamma_{11}(t) < x < \gamma_{14}(t), t \geq t_{\bar{H}}\} \end{aligned}$$

where $t_{\bar{Q}} = -\frac{u_0^+}{g^+}$, $t_{\bar{H}}, t_{\bar{H}}, \gamma_1, \gamma_6, \gamma_9, \gamma_{11}, \gamma_{14}$, are given in (2.33), (3.4), (3.5), (2.19), (2.19), (3.6) and (2.29) respectively. The regions R_i^{12} ($i = 1, 2, 3, 4, 7, 8, 10$) and the characteristics of the solution (3.11) are shown in Fig. 3.3.

Similar to the proof of Theorem 3.1, we can obtain the conclusions of Theorems 3.3 and 3.2 by integration along characteristics.

When $g^- < 0 < g^+$, the source term creates a “vacuum” when t is large. Compared with the case of shocks, the “vacuum” may issue from the origin for the problem of rarefaction waves. In the following theorem, we study the case that the source

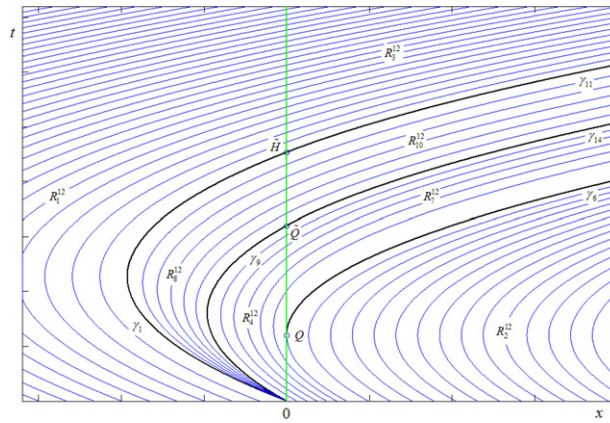


Fig. 3.3. The rarefaction wave $R_8^{12} \cup R_{10}^{12}$, $\{x = 0\}$ and $\{x = \gamma_6(t)\}$ are two weak discontinuities.

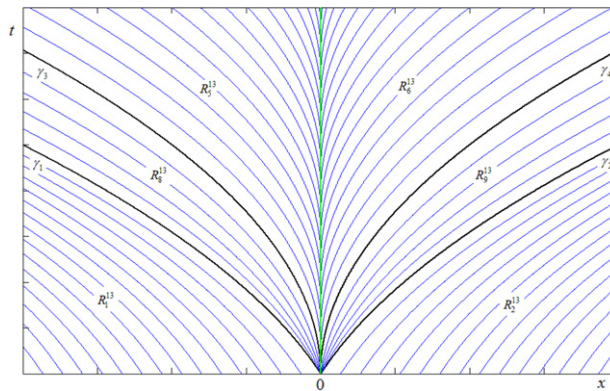


Fig. 3.4. Rarefaction wave $R_8^{13} \cup R_9^{13}$, R_5^{13} and R_6^{13} are “vacuum” regions.

term creates a “vacuum” immediately from the origin. To determine the solution in this “vacuum” region, first we obtain an approximate solution sequence to the initial value problem with the discontinuous source term being approximated by a continuous one, then the limit of this solution sequence can be regarded as the entropy solution to the original problem. The validity of this argument shall be verified by studying the uniqueness of entropy solutions in the next section.

Theorem 3.4. Suppose $u_0^- < 0 < u_0^+$ and $g^- < 0 < g^+$, then the problem (1.1) has a global entropy solution,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{13} \quad (i = 1, 2, 5, 6, 8, 9) \quad (3.12)$$

where $u_1, u_2, u_5, u_6, u_8, u_9$ are given by (2.2), (2.2), (2.34), (2.34), (3.7), (3.1) respectively and the regions R_i^{13} ($i = 1, 2, 5, 6, 8, 9$) are defined as

$$\begin{aligned} R_1^{13} &= \{x < \gamma_1(t), t \geq 0\} & R_2^{13} &= \{x > \gamma_2(t), t \geq 0\} \\ R_5^{13} &= \{\gamma_3(t) < x < 0, t \geq 0\} & R_6^{13} &= \{0 < x < \gamma_4(t), t \geq 0\} \\ R_8^{13} &= \{\gamma_1(t) < x < \gamma_3(t), t \geq 0\} & R_9^{13} &= \{\gamma_4(t) < x < \gamma_2(t), t \geq 0\} \end{aligned}$$

where $\gamma_1(t), \gamma_2(t)$ are given by (3.5), (3.2) respectively, and $\gamma_3(t), \gamma_4(t)$ denote

$$\gamma_3(t) = \frac{1}{2}g^-t^2 \quad \gamma_4(t) = \frac{1}{2}g^+t^2. \quad (3.13)$$

The regions R_i^{13} ($i = 1, 2, 5, 6, 8, 9$) and the characteristics of the solution (3.12) are shown in Fig. 3.4.

Proof. By the method of characteristics we can get the solution in R_1^{13} and R_2^{13} immediately. As shown in Fig. 3.4, using the method of characteristics, one cannot determine the solution u in the region $\{\gamma_1(t) < x < \gamma_2(t), t > 0\}$ directly from the data. In order to determine the entropy solution u in this region, we first use two sequences of continuous functions g^ε and u_0^ε to approximate g and u_0 respectively. Consider the following problem:

$$\begin{cases} u_t + uu_x = g^\varepsilon(x, t), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0^\varepsilon(x), & x \in \mathbb{R} \end{cases} \quad (3.14)$$

where

$$g^\varepsilon(x, t) = \begin{cases} g^- & x < -\varepsilon \\ -\frac{1}{\varepsilon}g^-x & -\varepsilon \leq x < 0 \\ \frac{1}{\varepsilon}g^+x & 0 < x < \varepsilon \\ g^+ & x > \varepsilon, \end{cases} \quad u_0^\varepsilon(x, t) = \begin{cases} u_0^- & x < -2\varepsilon \\ u_0^- \left(-\frac{x}{\varepsilon} - 1\right) & -2\varepsilon \leq x \leq -\varepsilon \\ 0 & -\varepsilon < x < \varepsilon \\ u_0^+ \left(\frac{x}{\varepsilon} - 1\right) & \varepsilon \leq x \leq 2\varepsilon \\ u_0^+ & x > 2\varepsilon \end{cases} \quad (3.15)$$

are continuous approximation of $g(x, t)$ and $u_0(x)$ respectively.

By a direct computation, we get the solution to (3.14) as follows

$$u^\varepsilon(x, t) = \begin{cases} g^-t + u_0^- & x < \frac{1}{2}g^-t^2 + u_0^-t - 2\varepsilon \\ g^-t + \frac{(2x - g^-t^2 + 2\varepsilon)u_0^-}{2(u_0^-t - \varepsilon)} & \frac{1}{2}g^-t^2 + u_0^-t - 2\varepsilon < x < \frac{1}{2}g^-t^2 - \varepsilon \\ -\sqrt{-g^-\varepsilon\hat{L}^2 + 2g^-\varepsilon + 2g^-x} & \frac{1}{2}g^-t^2 - \varepsilon < x < -\varepsilon \\ x\tilde{L}\sqrt{\frac{-g^-}{\varepsilon}} & -\varepsilon < x < 0 \\ x\tilde{R}\sqrt{\frac{g^+}{\varepsilon}} & 0 < x < \varepsilon \\ \sqrt{g^+\varepsilon\hat{R}^2 - 2g^+\varepsilon + 2g^+x} & \varepsilon < x < \frac{1}{2}g^+t^2 + \varepsilon \\ g^+t + \frac{(2x - g^+t^2 - 2\varepsilon)u_0^+}{2(u_0^+t + \varepsilon)} & \frac{1}{2}g^+t^2 + \varepsilon < x < \frac{1}{2}g^+t^2 + u_0^+t + 2\varepsilon \\ g^+t + u_0^+ & x > \frac{1}{2}g^+t^2 + u_0^+t + 2\varepsilon. \end{cases} \quad (3.16)$$

The solution (3.16) and its characteristics are shown in Fig. 3.5.

In (3.16), $\tilde{L} = \tilde{L}(t) = \tanh\left(\sqrt{\frac{-g^-}{\varepsilon}}t\right)$, $\tilde{R} = \tilde{R}(t) = \tanh\left(\sqrt{\frac{g^+}{\varepsilon}}t\right)$. For any fixed $(x, t) \in \{\frac{1}{2}g^-t^2 - \varepsilon < x < -\varepsilon, t > 0\}$, $\hat{L} = \hat{L}(t_0) = \tanh\left(\sqrt{\frac{-g^-}{\varepsilon}}t_0(x, t)\right)$ and $t_0(x, t)$ is the solution to the equation,

$$g^-t_0 = g^-t - \sqrt{-g^-\varepsilon\hat{L}(t_0) + \sqrt{-g^-\varepsilon(\hat{L}(t_0))^2 + 2g^-\varepsilon + 2g^-x}} \quad (3.17)$$

and for any fixed $(x, t) \in \{\varepsilon < x < \frac{1}{2}g^+t^2 + \varepsilon, t > 0\}$, $\hat{R} = \hat{R}(t_0) = \tanh\left(\sqrt{\frac{g^+}{\varepsilon}}t_0(x, t)\right)$ and t_0 is the solution of the equation:

$$g^+t_0 = g^+t + \sqrt{g^+\varepsilon\hat{R}(t_0) - \sqrt{g^+\varepsilon(\hat{R}(t_0))^2 - 2g^+\varepsilon + 2g^+x}}. \quad (3.18)$$

In Remark 3.1, we will show the uniqueness of the solutions of (3.17) and (3.18).

From $\hat{L} = \tanh\left(\sqrt{\frac{-g^-}{\varepsilon}}t_0\right)$ and $\hat{R} = \tanh\left(\sqrt{\frac{g^+}{\varepsilon}}t_0\right)$ we know \hat{L} and \hat{R} are both bounded, so as $\varepsilon \rightarrow 0$, u^ε converges pointwisely to the function given in (3.12). Thus, we obtain the entropy solution u in the region $\{\gamma_1(t) < x < \gamma_2(t), t > 0\}$. \square

Remark 3.1. It can be shown that both (3.17) and (3.18) have unique solutions. For example, for the Eq. (3.18), let

$$F(t_0) = g^+t_0 - g^+t - \sqrt{g^+\varepsilon\hat{R}} + \sqrt{g^+\varepsilon\hat{R}^2 - 2g^+\varepsilon + 2g^+x}.$$

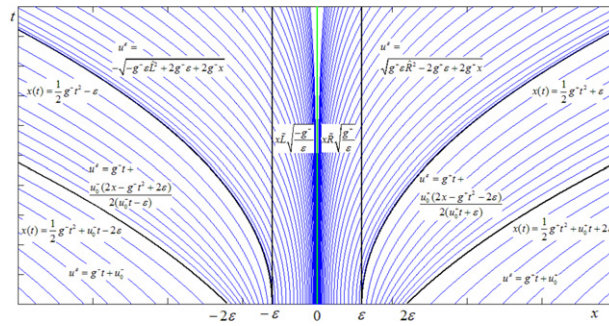


Fig. 3.5. The approximate solution (3.16) and its characteristics.

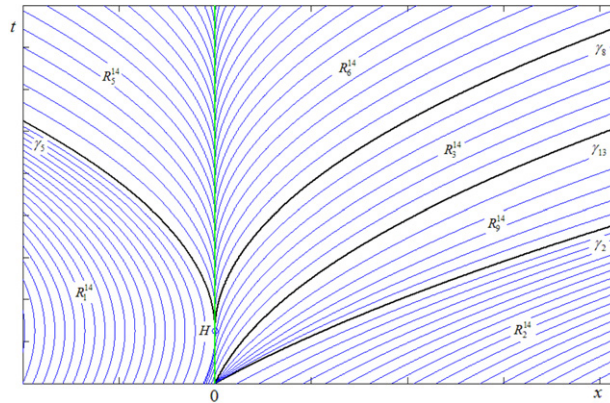


Fig. 3.6. Rarefaction wave R_9^{14} , and the vacuum regions appear from the point H .

It is easy to have,

$$\begin{aligned} (1) & F(0) = -g^+ t + \sqrt{2g^+(x - \varepsilon)} < 0, \\ (2) & F(t_0) > g^+(t_0 - t) \rightarrow +\infty \text{ as } t \rightarrow +\infty, \\ (3) & F'(t_0) = g^+ + g^+ \cosh^{-2} \left(\sqrt{\frac{g^+}{\varepsilon}} t_0 \right) \left(\frac{\sqrt{g^+ \varepsilon \hat{R}}}{\sqrt{g^+ \varepsilon \hat{R}^2 - 2g^+ \varepsilon + 2g^+ x}} - 1 \right) \\ & \geq g^+ \left(1 - \cosh^{-2} \left(\sqrt{\frac{g^+}{\varepsilon}} t_0 \right) \right) > 0, \quad \forall t_0 > 0. \end{aligned}$$

Thus, for all $(x, t) \in \{\varepsilon < x < \frac{1}{2} g^+ t^2 + \varepsilon, t > 0\}$, the equation $F(t_0) = 0$ has a unique root t_0 in $(0, +\infty)$.

Remark 3.2. An interesting question is that if we use another pair of $(g^\varepsilon, u_0^\varepsilon)$ to approximate (g, u_0) can we get the same solution as (3.12)? Theorem 4.1 in Section 4 will give a positive answer to this question.

In the following theorem, we study the case that the discontinuous source term creates a “vacuum” some time later, more precisely, the vacuum appears from the point H being given in (2.33) (see Fig. 3.6). This phenomenon happens when $g^- < 0 < g^+$ and the rarefaction wave is located in one side of the discontinuity of the source term for small t .

Theorem 3.5. Suppose $g^- < 0 < g^+$ and $0 < u_0^- < u_0^+$, then the problem (1.1) has an entropy solution containing a rarefaction wave and a “vacuum” region,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{14} \quad (i = 1, 2, 3, 5, 6, 9) \quad (3.19)$$

where $u_1, u_2, u_3, u_5, u_6, u_9$ are given by (2.2), (2.2), (2.1), (2.34), (2.34), (3.1) respectively and the regions R_i^{14} ($i = 1, 2, 3, 5, 6, 9$) are defined as

$$\begin{aligned} R_1^{14} &= \{x < 0, 0 \leq t < t_H\} \cup \{x < \gamma_5(t), t \geq t_H\} \\ R_2^{14} &= \{x > \gamma_2(t), t \geq 0\} \\ R_3^{14} &= \{0 < x < \gamma_{13}(t), 0 \leq t < t_H\} \cup \{\gamma_8(t) < x < \gamma_{13}(t), t \geq t_H\} \end{aligned}$$

$$\begin{aligned} R_5^{14} &= \{\gamma_5(t) < x < 0, t \geq t_H\} \\ R_6^{14} &= \{0 < x < \gamma_6(t), t \geq t_H\} \\ R_9^{14} &= \{\gamma_{13}(t) < x < \gamma_2(t), t > 0\} \end{aligned}$$

shown in Fig. 3.6, where t_H , γ_2 , γ_5 , γ_8 and γ_{13} are given by (2.33), (3.2), (2.35), (2.35) and (3.2) respectively.

3.2. Bend of a rarefaction wave with a shock

In this subsection, we consider $g^- > g^+$, which implies $g^- > 0$ obviously from the assumption $g^+ + g^- > 0$. Under this condition the discontinuous source term creates a shock for large t . This is because when $g^- > 0 > g^+$ or $g^- > g^+ > 0$, the solution u in the second quadrant increases faster than that in the first quadrant, so at some time the value of u in the left side of t -axis will be larger than its value in the right side, which yields a shock.

In particular, when $g^- > 0 > g^+$ and $g^- - g^+$ is sufficiently large, the source term can create a shock immediately from the origin. This case is described in the following theorem.

Theorem 3.6. Suppose $u_0^+ > 0 > u_0^-$, $g^- > 0 > g^+$ and $\frac{u_0^-}{u_0^+} > \frac{3g^-}{2g^+ - g^-}$, then the problem (1.1) has a piecewise \mathcal{C}^1 solution,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{15} \quad (i = 1, 2, 3, 8, 9, 10) \quad (3.20)$$

where $u_1, u_2, u_3, u_8, u_9, u_{10}$ are given by (2.2), (2.2), (2.1), (3.7), (3.1), (3.8) respectively and the regions R_i^{15} ($i = 1, 2, 3, 8, 9, 10$) are defined as

$$\begin{aligned} R_1^{15} &= \{x < \gamma_1(t), 0 \leq t < t_{\bar{H}}\} \cup \{x < 0, t \geq t_{\bar{H}}\} \\ R_2^{15} &= \{x > \gamma_2(t), 0 \leq t < t_B\} \cup \{x > \varphi(t), t \geq t_B\} \\ R_3^{15} &= \{0 < x < \gamma_{11}(t), t_{\bar{H}} \leq t < t_P\} \cup \{0 < x < \varphi(t), t \geq t_P\} \\ R_8^{15} &= \{\gamma_1(t) < x < 0, 0 \leq t < t_{\bar{H}}\} \\ R_9^{15} &= \{\varphi(t) < x < \gamma_2(t), 0 \leq t < t_B\} \\ R_{10}^{15} &= \{0 < x < \varphi(t), 0 \leq t < t_{\bar{H}}\} \cup \{\gamma_{11}(t) < x < \varphi(t), t_{\bar{H}} \leq t < t_P\} \end{aligned}$$

where $t_{\bar{H}}, \gamma_1, \gamma_2, \gamma_{11}$ are given by (3.4), (3.5), (3.2), (3.6) respectively. For the solution (3.20), the fan of rarefaction wave is $R_8^{15} \cup R_9^{15}$, in which there appears a new shock with the front $x = \varphi(t)$ being given by

$$\varphi(t) = \begin{cases} x_1(t) & 0 \leq t < t_P \\ x_2(t) & t_P \leq t < t_B \\ x_3(t) & t \geq t_B \end{cases}$$

where $x_1(t) = \frac{(g^+ - 2g^-)(g^+ + g^-)}{18(g^+ - g^-)} t^2$ and $x_2(t), x_3(t)$ are determined by

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_3(x_2(t), t) + u_9(x_2(t), t)) & t > t_P \\ x_2(t_P) = x_P \end{cases} \quad (3.21)$$

and

$$\begin{cases} x_3'(t) = \frac{1}{2}(u_3(x_3(t), t) + u_2(x_3(t), t)) & t > t_B \\ x_3(t_B) = x_B \end{cases} \quad (3.22)$$

respectively, with $P = (t_P, x_P)$ and $B = (t_B, x_B)$ are the intersection points of the shock front $x = \varphi(t)$ with the rarefaction tails $x = \gamma_{11}(t)$ and $x = \gamma_2(t)$ respectively. The behavior of the solution is described in Fig. 3.7.

Proof. By the method of characteristics we can deduce the solution $u(x, t)$ in the regions R_1^{15} and R_2^{15} immediately. The rarefaction wave in regions R_8^{15} and R_9^{15} can be obtained by the same method as that used in the proof of the Theorem 3.1.

Under the assumption of the discontinuous source term, we know that the characteristics starting from $\{x \leq 0, t = 0\}$ will cross t -axis and generate a shock wave on its right side. By a direct calculation, we obtain the value of u in the regions R_3^{15} and R_{10}^{15} , and the common boundary of these two regions is $x = \gamma_{11}(t)$. From the Rankine–Hugoniot jump condition we know that the shock front $x_1(t)$ satisfies,

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_{10}(x_1(t), t) + u_9(x_1(t), t)) \\ x_1(0) = 0. \end{cases} \quad (3.23)$$

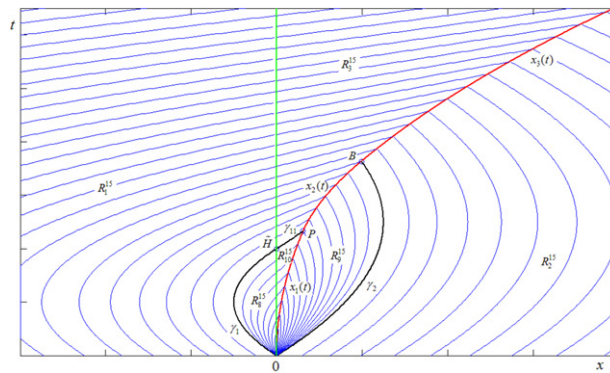


Fig. 3.7. The rarefaction wave fan is $R_8^{15} \cup R_9^{15}$, and the red curve is the shock front. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

It can be checked that $x_1(t) = \frac{(g^+ - 2g^-)(g^+ + g^-)}{18(g^+ - g^-)} t^2$ is the solution of (3.23), so $x = x_1(t)$ passes $P = \left(\frac{(g^+ + g^-)(g^+ - g^-)(g^+ - 2g^-)}{2(2g^+ - g^-)^2(g^-)^2} (u_0^-)^2, \frac{3(g^- - g^+)}{(2g^+ - g^-)g^-} u_0^- \right)$. As $\frac{u_0^-}{u_0^+} > \frac{3g^-}{2g^+ - g^-}$, the point P should be on the left side of the curve $x = \gamma_2(t)$. Thus, the region R_9^{15} should be filled by the rarefaction wave $u(x, t) = u_9(x, t)$. So, from the Rankine–Hugoniot jump condition we know that $x_2(t)$ and $x_3(t)$ satisfy (3.21) and (3.22) respectively. Thus, (3.20) holds. \square

Remark 3.3. If $\frac{u_0^-}{u_0^+} = \frac{3g^-}{2g^+ - g^-}$, then the curve $x = \gamma_2(t)$ passes the point P and $x = x_2(t)$ disappears. If u_0^+ continues to decrease such that $\frac{u_0^-}{u_0^+} < \frac{3g^-}{2g^+ - g^-}$, then the curve $x = \gamma_2(t)$ will intersect with $x = x_1(t)$. So the point P will get close to \tilde{H} . Now let us consider the limit case $u_0^+ = 0$.

- (1) If $2g^+ + g^- \geq 0$, then $\frac{1}{2}g^-t + \frac{x}{t} + g^+t \geq 0$ on $\{x = 0, 0 < t \leq t_H\}$. So the shock front will start from the origin and get into the right side of t -axis.
- (2) If $2g^+ + g^- < 0$, then $\frac{1}{2}g^-t + \frac{x}{t} + g^+t < 0$ on $\{x = 0, 0 < t \leq t_H\}$. So the shock front will start from the origin and get into the left side of the t -axis. Thus, if u_0^+ is small enough, we know the shock wave will get into the left side of t -axis for small t .

Remark 3.4. It can be proven that even if the condition $\frac{u_0^-}{u_0^+} > \frac{3g^-}{2g^+ - g^-}$ given in Theorem 3.6 does not hold, the shock front will be located at the positive x eventually when t is large enough. Actually, if this assertion was not true, which means that the shock front $\varphi(t) \leq 0$ as $t > t_0$ for a large t_0 , then it can be determined by solving the following problem:

$$\begin{cases} \varphi'(t) = \frac{1}{2}(u_1(\varphi(t), t) + u_4(\varphi(t), t)) \\ x(t_0) = x_0 \end{cases}$$

with $t_0 > -\frac{2u_0^+}{g^+}$, $x_0 < 0$ and u_4 is given by (2.14). Considering $g^- > 0 > g^+$, $g^+ + g^- > 0$ and $\varphi(t) \leq 0$ as $t > t_0$, we have

$$\begin{aligned} \varphi'(t) &\geq \frac{1}{2} \left(g^-t + u_0^- + g^-t + u_0^+ + \frac{g^+ - g^-}{g^- - 2g^+} \left(u_0^+ - g^+t + g^-t + \sqrt{(g^+t + u_0^+)^2} \right) \right) \\ &= (g^+ + g^-)t + u_0^+ + u_0^- \end{aligned}$$

which immediately implies $\varphi(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, by noting $g^+ + g^- > 0$. Thus, the shock wave will get into the right side of the upper half-plane when t is large.

The following result describes the case in which the source term creates a shock after some time. This happens when $g^- > g^+$ and the rarefaction wave is located on one side of the discontinuity of the source term for small t .

Theorem 3.7. Suppose $u_0^- < u_0^+ < 0$ and $g^- > 0 > g^+$, then the problem (1.1) has a piecewise \mathcal{C}^1 solution,

$$u(x, t) = u_i(x, t), \quad \text{as } (x, t) \in R_i^{16} \quad (i = 1, 2, 3, 9) \quad (3.24)$$

where u_1, u_2, u_3, u_9 are given by (2.2), (2.2), (2.1), (3.1) respectively and the regions R_i^{16} ($i = 1, 2, 3, 9$) are defined as

$$\begin{aligned} R_1^{16} &= \{x < 0, t \geq 0\} \\ R_2^{16} &= \{x > \gamma_2(t), 0 \leq t < t_B\} \cup \{x > \varphi(t), t \geq t_B\} \end{aligned}$$

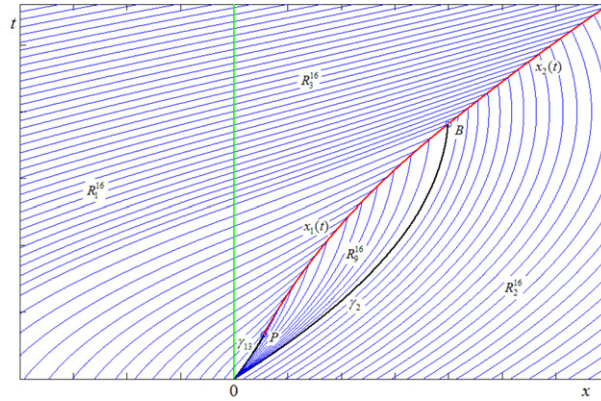


Fig. 3.8. The shock front $x_1(t)$ forms from the point P and $x = \gamma_{13}(t)$ is a weak discontinuity.

$$\begin{aligned} R_3^{16} &= \{0 < x < \gamma_{13}(t), 0 \leq t < t_P\} \cup \{0 < x < \varphi(t), t \geq t_P\} \\ R_9^{16} &= \{\gamma_{13}(t) < x < \gamma_2(t), 0 \leq t < t_P\} \cup \{\varphi(t) < x < \gamma_2(t), t_P \leq t < t_B\} \end{aligned}$$

where $\gamma_2(t)$, $\gamma_{13}(t)$ are given by (3.2) and the shock front $x = \varphi(t)$ is defined as

$$\varphi(t) = \begin{cases} x_1(t) & t_P \leq t < t_B \\ x_2(t) & t \geq t_B \end{cases}$$

in which $x_1(t)$ and $x_2(t)$ are determined by

$$\begin{cases} x_1'(t) = \frac{1}{2}(u_3(x_1(t), t) + u_9(x_1(t), t)) & t > t_P \\ x_1(t_P) = x_P \end{cases}$$

and

$$\begin{cases} x_2'(t) = \frac{1}{2}(u_2(x_2(t), t) + u_2(x_2(t), t)) & t > t_B \\ x_2(t_B) = x_B \end{cases}$$

respectively, with $(t_P, x_P) = (\frac{2g^- - g^+}{(2(g^+ - g^-)^2}(u_0^-)^2, \frac{u_0^-}{g^- - g^+}))$ and $B = (t_B, x_B)$ being the intersection point of $x = x_1(t)$ with the curve $x = \gamma_2(t)$. The regions R_i^{16} ($i = 1, 2, 3, 9$) and the characteristics of the solution (3.24) are shown in Fig. 3.8.

4. Uniqueness and stability of entropy solutions

In this section, we will prove the uniqueness of the entropy solution and the stability with respect to the initial data and source terms. The idea is to provide an estimate of L^1 distance between any two solutions of (1.1) in the space \mathcal{A} given by (4.10). This estimate implies that the entropy solution of (1.1) is unique in this space. As a consequence, we obtain that the solutions constructed in Sections 2 and 3 are unique and stable. In particular, the solutions constructed in previous sections for the cases that some “vacuum” regions appear make sense and are independent of the ways we constructed.

First, we derive an entropy inequality for the original problem as follows.

Lemma 4.1. Suppose $f \in C^2(\mathbb{R})$, $f'' > 0$ on \mathbb{R} and $g, u \in L_{loc}^\infty(\mathbb{R} \times [0, \infty))$, $u_0 \in L_{loc}^\infty(\mathbb{R})$. Let u is a weak solution of

$$\begin{cases} u_t + f(u)_x = g & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x) & x \in \mathbb{R}. \end{cases} \quad (4.1)$$

In addition, u is C^1 except on $\{x = \gamma_j(t)\}_{j=1}^N$ and satisfies the Lax entropy condition on these curves,

$$f'(u_i^-(t)) \geq \gamma_i'(t) \geq f'(u_i^+(t)), \quad \forall t \geq 0, 1 \leq i \leq N \quad (4.2)$$

where $u_i^-(t) = \lim_{\tau \rightarrow 0^+} u(\gamma_i(t) - \tau, t)$ and $u_i^+(t) = \lim_{\tau \rightarrow 0^+} u(\gamma_i(t) + \tau, t)$. For any fixed $k \in \mathbb{R}$, define $\eta(u) \doteq |u - k|$, $q(u) \doteq (f(u) - f(k))\text{sgn}(u - k)$. Then, we have the following entropy inequality,

$$\gamma_i'(t)(\eta(u_i^+(t)) - \eta(u_i^-(t))) \geq q(u_i^+(t)) - q(u_i^-(t)), \quad \forall t \geq 0, 1 \leq i \leq N \quad (4.3)$$

$$\iint_{t \geq 0} \{\eta(u)\phi_t + q(u)\phi_x + \text{sgn}(u - k)g\phi\} dx dt \geq 0 \quad (4.4)$$

for all non-negative function $\phi \in C_c^1(\mathbb{R} \times (0, \infty))$.

Proof. This result is classical, e.g. [12,13]. For completeness, we recall the main steps of the proof.

First, we prove the inequality (4.3). From $f'' > 0$ and (4.2) we know f' is a strictly increasing function and $u_i^-(t) \geq u_i^+(t)$, $\forall t \geq 0$, $1 \leq i \leq N$. For any fixed $t \geq 0$ and $1 \leq i \leq N$, if $u_i^-(t) = u_i^+(t)$ then (4.3) holds obviously. So, we assume $u_i^-(t) > u_i^+(t)$ for some i , there are three cases:

- (1) $k \geq u_i^-(t) > u_i^+(t)$,
- (2) $u_i^-(t) > u_i^+(t) \geq k$,
- (3) $u_i^-(t) > k > u_i^+(t)$.

For the above Case 1 and Case 2, as γ_i satisfies the Rankine–Hugoniot jump condition, it can be easily verified that (4.3) holds, so we only need consider the Case 3. For a fixed t , for simplicity, we use u_i^- and u_i^+ to denote $u_i^-(t)$ and $u_i^+(t)$ respectively. Let $\alpha \doteq \frac{k-u_i^+}{u_i^- - u_i^+}$, then $k = \alpha u_i^- + (1 - \alpha)u_i^+$ and $\alpha \in (0, 1)$ for the Case 3. From $f'' > 0$ we know f is convex which implies

$$\alpha f(u_i^-) + (1 - \alpha)f(u_i^+) \geq f(\alpha u_i^- + (1 - \alpha)u_i^+). \quad (4.5)$$

From (4.5), $k = \alpha u_i^- + (1 - \alpha)u_i^+$, $u_i^- > u_i^+$ and the obvious identity

$$(\alpha u_i^- + (1 - \alpha)u_i^+)(f(u_i^-) - f(u_i^+)) + u_i^- f(u_i^+) - u_i^+ f(u_i^-) = (u_i^- - u_i^+)(\alpha f(u_i^-) + (1 - \alpha)f(u_i^+))$$

we obtain

$$2k(f(u_i^-) - f(u_i^+)) + 2u_i^- f(u_i^+) - 2u_i^+ f(u_i^-) \geq 2(u_i^- - u_i^+)f(k)$$

which is equivalent to

$$\frac{f(u_i^-) - f(u_i^+)}{u_i^- - u_i^+}(2k - u_i^- - u_i^+) \geq 2f(k) - f(u_i^-) - f(u_i^+). \quad (4.6)$$

Considering that γ_i satisfies the Rankine–Hugoniot jump condition and $u_i^- > k > u_i^+$, from (4.6) we obtain the inequality (4.3) immediately.

Now we study the inequality (4.4).

Define $\Omega^+ \doteq \{(x, t) \in \mathbb{R} \times [0, \infty) | u(x, t) - k \geq 0\}$, and $\Omega^- \doteq \{(x, t) \in \mathbb{R} \times [0, \infty) | u(x, t) - k < 0\}$. Because u is \mathcal{C}^1 except on $\{x = \gamma_j(t)\}_{j=1}^N$ and $\eta(u) = q(u) = 0$ on $\partial\Omega^+$ and $\partial\Omega^-$, from the Green formula we have,

$$\begin{aligned} \iint_{\Omega^+} \eta(u)\phi_t + q(u)\phi_x dxdt &= - \iint_{\Omega^+} g\phi dxdt + \sum_{i=1}^N \int_{a_i^+}^{b_i^+} \{\gamma_i'(t)(\eta(u_i^+(t)) - \eta(u_i^-(t))) \\ &\quad - (q(u_i^+(t)) - q(u_i^-(t)))\} \phi(\gamma_i(t), t) dt \end{aligned} \quad (4.7)$$

where a_i^+ , b_i^+ are the t -coordinates of the intersection points of $\text{supp } \phi \cap \Omega_i^+$ and $\{x = \gamma_i(t)\}$, and similarly we obtain,

$$\begin{aligned} \iint_{\Omega^-} \eta(u)\phi_t + q(u)\phi_x dxdt &= \iint_{\Omega^-} g\phi dxdt + \sum_{i=1}^N \int_{a_i^-}^{b_i^-} \{\gamma_i'(t)(\eta(u_i^-(t)) - \eta(u_i^+(t))) \\ &\quad - (q(u_i^-(t)) - q(u_i^+(t)))\} \phi(\gamma_i(t), t) dt \end{aligned} \quad (4.8)$$

where a_i^- , b_i^- are the t -coordinates of the intersection points of $\text{supp } \phi \cap \Omega_i^-$ and $\{x = \gamma_i(t)\}$.

Adding (4.7) and (4.8), we obtain,

$$\begin{aligned} \iint_{t \geq 0} \eta(u)\phi_t + q(u)\phi_x + \text{sgn}(u - k)g\phi dxdt \\ = \sum_{i=1}^N \int_{a_i}^{b_i} \{\gamma_i'(t)(\eta(u_i^+(t)) - \eta(u_i^-(t))) - (q(u_i^+(t)) - q(u_i^-(t)))\} \phi(\gamma_i(t), t) dt \end{aligned} \quad (4.9)$$

where a_i , b_i are the t -coordinates of the intersection points of $\text{supp } \phi$ and $\{x = \gamma_i(t)\}$. Thus, from (4.3) and (4.9) we conclude the inequality (4.4). \square

Now, we define the space \mathcal{A} ,

$$\begin{aligned} \mathcal{A} &= \{u \in \mathbf{L}_{\text{loc}}^\infty(\mathbb{R} \times [0, \infty)) | \text{the map } t \mapsto u(\cdot, t) \text{ is continuous from } [0, \infty) \text{ into } \mathbf{L}_{\text{loc}}^1(\mathbb{R}); \\ &\quad \forall T > 0, \exists M(T) > 0 \text{ s.t. } |u(x, t)| \leq M(T), \forall (x, t) \in \mathbb{R} \times [0, T]; u \text{ is } \mathcal{C}^1 \text{ except} \\ &\quad \text{on finite curves on which } u \text{ is continuous or satisfies the Lax entropy condition}\}. \end{aligned} \quad (4.10)$$

By this definition, first we have that the solution obtained in (3.12) belongs to \mathcal{A} . In fact, we only need to prove the map $t \mapsto u(\cdot, t)$ is continuous from $[0, \infty)$ to $\mathbf{L}_{\text{loc}}^1(\mathbb{R})$ and this can be shown by the following fact:

$$\lim_{t \rightarrow 0^+} \left\{ \int_{\frac{1}{2}g^-t^2 + u_0^-t}^{\frac{1}{2}g^-t^2} \left(\frac{1}{2}g^-t + \frac{x}{t} \right) dx + \int_{\frac{1}{2}g^-t^2}^0 (-\sqrt{2g^-x}) dx \right. \\ \left. + \int_0^{\frac{1}{2}g^+t^2} \sqrt{2g^+x} dx + \int_{\frac{1}{2}g^+t^2}^{\frac{1}{2}g^+t^2 + u_0^+t} \left(\frac{1}{2}g^+t + \frac{x}{t} \right) dx \right\} = 0.$$

Therefore, by the following theorem the solution (3.12) is irrelevant to the selection of the sequences u_0^ε and g^ε given by (3.15). Theorem 4.1 is a generalization of the classical result of Kruzhkov in [14].

Theorem 4.1. Suppose f, g^1, g^2 and u_0^1, u_0^2 satisfy the conditions given in Lemma 4.1. Let $u^1, u^2 \in \mathcal{A}$ be the solutions of (4.1) with $(g, u_0) = (g^1, u_0^1)$ and (g^2, u_0^2) respectively, and satisfying $u_0^1 - u_0^2 \in \mathbf{L}^1(\mathbb{R})$, $g^1 - g^2 \in \mathbf{L}^1(\mathbb{R} \times [0, T])$ for all $T > 0$. Then, for all $\tau > 0$ we have,

$$\int_{-\infty}^{+\infty} |u^1(x, \tau) - u^2(x, \tau)| dx \leq \int_{-\infty}^{+\infty} |u^1(x, 0) - u^2(x, 0)| dx + \int_0^\tau \int_{-\infty}^{+\infty} |g^1(x, t) - g^2(x, t)| dx dt. \quad (4.11)$$

Proof. For any fixed $R > 0$ and $\tau > 0$, from $u^1, u^2 \in \mathcal{A}$ we know there exists $M > 0$ such that

$$|u^1(x, t)| \leq M, \quad |u^2(x, t)| \leq M, \quad \text{for all } (x, t) \in \mathbb{R} \times [0, \tau + 1]. \quad (4.12)$$

Since $f \in \mathcal{C}^2(\mathbb{R})$, of course f is locally Lipschitz continuous, there exists $L > 0$ such that

$$|f(\omega) - f(\omega')| \leq L|\omega - \omega'|, \quad \text{for all } \omega, \omega' \in [-M, M]. \quad (4.13)$$

For any fixed $\tau_0 \in (0, \tau)$, we will prove the following result firstly.

$$\int_{|x| \leq R} |u^1(x, \tau) - u^2(x, \tau)| dx \leq \int_{|x| \leq R+L\tau} |u^1(x, \tau_0) - u^2(x, \tau_0)| dx \\ + \int_{\tau_0}^\tau \int_{-R-L(\tau-t)}^{R+L(\tau-t)} |g^1(x, t) - g^2(x, t)| dx dt. \quad (4.14)$$

Given any constants $k, k' \in \mathbb{R}$ and any smooth function $\phi = \phi(x, s, y, t) \geq 0$ with compact support, from (4.4) in the Lemma 4.1 we have

$$\iint \left\{ |u^1(x, s) - k| \phi_s(x, s, y, t) + \text{sgn}(u^1(x, s) - k) (f(u^1(x, s)) - f(k)) \phi_x(x, s, y, t) \right. \\ \left. + \text{sgn}(u^1(x, s) - k) g^1(x, s) \phi(x, s, y, t) \right\} dx ds \geq 0 \quad (4.15)$$

and

$$\iint \left\{ |u^2(y, t) - k'| \phi_t(x, s, y, t) + \text{sgn}(u^2(y, t) - k') (f(u^2(y, t)) - f(k')) \phi_y(x, s, y, t) \right. \\ \left. + \text{sgn}(u^2(y, t) - k') g^2(y, t) \phi(x, s, y, t) \right\} dy dt \geq 0. \quad (4.16)$$

Set $k = u^2(y, t)$ in (4.15) and integrate with respect to y, t , and set $k' = u^1(x, s)$ in (4.16) and integrate with respect to x, s . Adding these two results, we obtain

$$\iiint \left\{ |u^1(x, s) - u^2(y, t)| (\phi_s + \phi_t)(x, s, y, t) \right. \\ \left. + [f(u^1(x, s)) - f(u^2(y, t))] (\phi_x + \phi_y)(x, s, y, t) \text{sgn}(u^1(x, s) - u^2(y, t)) \right. \\ \left. + \text{sgn}(u^1(x, s) - u^2(y, t)) [g^1(x, s) - g^2(y, t)] \phi(x, s, y, t) \right\} dx dy ds dt \geq 0. \quad (4.17)$$

Now we use a sequence of functions $\{\delta_h\}_{h \geq 1}$ to approximate the Dirac measure. More precisely, let $\tilde{\delta} : \mathbb{R} \mapsto [0, 1]$ be a \mathcal{C}^∞ function such that

$$\int_{-\infty}^{+\infty} \tilde{\delta}(z) dz = 1, \quad \tilde{\delta}(z) = 0 \text{ for all } z \notin [-1, 1]$$

and define

$$\delta_h(z) = h\tilde{\delta}(hz), \quad \alpha_h(z) = \int_{-\infty}^z \delta_h(s) ds. \quad (4.18)$$

Considering any non-negative smooth function $\psi = \psi(X, T)$ whose support is a compact subset of the open half plane where $T > 0$, and define

$$\phi(x, s, y, t) = \psi\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{s-t}{2}\right).$$

A direct computation yields

$$(\phi_s + \phi_t)(x, s, y, t) = \psi_T\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{s-t}{2}\right)$$

and

$$(\phi_x + \phi_y)(x, s, y, t) = \psi_X\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{s-t}{2}\right).$$

For h sufficiently large, the support of ϕ is contained in the set where $s > 0, t > 0$. From (4.17) it thus follows that:

$$\begin{aligned} & \iiint \delta_h\left(\frac{x-y}{2}\right) \delta_h\left(\frac{s-t}{2}\right) \left\{ |u^1(x, s) - u^2(y, t)| \psi_T\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \right. \\ & \quad + [f(u^1(x, s)) - f(u^2(y, t))] \operatorname{sgn}(u^1(x, s) - u^2(y, t)) \psi_X\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \\ & \quad \left. + [g^1(x, s) - g^2(y, t)] \operatorname{sgn}(u^1(x, s) - u^2(y, t)) \psi\left(\frac{x+y}{2}, \frac{s+t}{2}\right) \right\} dx dy ds dt \geq 0. \end{aligned} \quad (4.19)$$

We now compute the left hand side of (4.19) as $h \rightarrow \infty$. Using the variables

$$X = \frac{x+y}{2}, \quad Y = \frac{x-y}{2}, \quad T = \frac{s+t}{2}, \quad S = \frac{s-t}{2}$$

the inequality (4.19) becomes

$$\begin{aligned} & \iiint \delta_h(Y) \delta_h(S) \left\{ |u^1(X+Y, T+S) - u^2(X-Y, T-S)| \psi_T(X, T) \right. \\ & \quad + [f(u^1(X+Y, T+S)) - f(u^2(X-Y, T-S))] \operatorname{sgn}(u^1(X+Y, T+S) \\ & \quad - u^2(X-Y, T-S)) \psi_X(X, T) + [g^1(X+Y, T+S) - g^2(X-Y, T-S)] \\ & \quad \left. \times \operatorname{sgn}(u^1(X+Y, T+S) - u^2(X-Y, T-S)) \psi(X, T) \right\} dX dY dS dT \geq 0. \end{aligned} \quad (4.20)$$

Letting $h \rightarrow \infty$ in (4.20) and renaming the variables X, T , we thus obtain

$$\begin{aligned} & \iint \left\{ |u^1(x, t) - u^2(x, t)| \psi_t(x, t) + [f(u^1(x, t)) - f(u^2(x, t))] \operatorname{sgn}(u^1(x, t) - u^2(x, t)) \psi_x(x, t) \right. \\ & \quad \left. + (g^1(x, t) - g^2(x, t)) \operatorname{sgn}(u^1(x, t) - u^2(x, t)) \psi(x, t) \right\} dx dt \geq 0. \end{aligned} \quad (4.21)$$

Now we construct a smooth function ψ by setting

$$\psi(x, t) = [\alpha_h(t - \tau_0) - \alpha_h(t - \tau)] [1 - \alpha_h(|x| - R - L(\tau - t))].$$

Recall that α_h was defined at (4.18), so that $\alpha'_h = \delta_h \geq 0$. Using (4.21) with this particular test function ψ , we obtain

$$\iint |u^1(x, t) - u^2(x, t)| [\delta_h(t - \tau_0) - \delta_h(t - \tau)] [1 - \alpha_h(|x| - R - L(\tau - t))] dx dt$$

$$\begin{aligned}
& + \iint (g^1(x, t) - g^2(x, t)) \operatorname{sgn}(u^1(x, t) - u^2(x, t)) [\alpha_h(t - \tau_0) - \alpha_h(t - \tau)] \\
& \times [1 - \alpha_h(|x| - R - L(\tau - t))] dx dt \\
& \geq \iint \left\{ \operatorname{sgn}(x) [f(u^1(x, t)) - f(u^2(x, t))] \cdot \operatorname{sgn}(u^1(x, t) - u^2(x, t)) \right. \\
& \left. + L|u^1(x, t) - u^2(x, t)| \right\} [\alpha_h(t - \tau_0) - \alpha_h(t - \tau)] \cdot \delta_h(|x| - R - L(\tau - t)) dx dt.
\end{aligned} \tag{4.22}$$

By (4.12) and (4.13) we have $|f(u^1) - f(u^2)| \leq L|u^1 - u^2|$. Moreover, (2.19) yields $\alpha_h(t - \tau_0) - \alpha_h(t - \tau) \geq 0$, $\alpha'_h = \delta_h \geq 0$. Hence, from (4.22) we have

$$\begin{aligned}
& \iint (g^1(x, t) - g^2(x, t)) \operatorname{sgn}(u^1(x, t) - u^2(x, t)) [\alpha_h(t - \tau_0) - \alpha_h(t - \tau)] \cdot [1 - \alpha_h(|x| - R - L(\tau - t))] dx dt \\
& + \iint |u^1(x, t) - u^2(x, t)| [\delta_h(t - \tau_0) - \delta_h(t - \tau)] \cdot [1 - \alpha_h(|x| - R - L(\tau - t))] dx dt \\
& \geq 0.
\end{aligned} \tag{4.23}$$

Recalling that the map $t \mapsto u^i(\cdot, t)$ is continuous from $[0, \infty)$ into $L^1_{\text{loc}}(\mathbb{R})$ for $i = 1, 2$, we now let $h \rightarrow \infty$ in (4.23) and obtain (4.14). As $\tau_0 \in (0, \tau)$ is arbitrary, by the Lebesgue Dominated Convergence Theorem we know (4.14) still holding at $\tau_0 = 0$,

$$\int_{|x| \leq R} |u^1(x, \tau) - u^2(x, \tau)| dx \leq \int_{|x| \leq R+L\tau} |u^1(x, 0) - u^2(x, 0)| dx + \int_0^\tau \int_{-R-L(\tau-t)}^{R+L(\tau-t)} |g^1(x, t) - g^2(x, t)| dx dt. \tag{4.24}$$

Because $R > 0$ is arbitrary, letting $R \rightarrow +\infty$ in (4.24) we conclude (4.11). \square

From Theorem 4.1 we know the solution of (4.1) is stable and unique in \mathcal{A} .

Acknowledgments

Beixiang Fang is supported in part by the NNSF of China under Grant Nos 10801096 and 11031001. Pingfan Tang is supported in part by the Shanghai Jiao Tong University Innovation Fund for Postgraduates and the NNSF of China under Grant Nos 10801096, 10971134 and 11031001. Ya-Guang Wang is supported in part by the NNSF of China under Grant Nos 10971134 and 11031001.

References

- [1] D. Mihalas, B.W. Mihalas, Foundations of Radiation Hydrodynamics, Oxford University Press, New York, Oxford, 1984.
- [2] G.C. Pomraning, The Equations of Radiation Hydrodynamics, Pergamon Press, 1973.
- [3] X.H. Zhong, S. Jiang, Local existence and finite-time blow-up in multi-dimensional radiation hydrodynamics, J. Math. Fluid Mech. 9 (2007) 543–564.
- [4] P. Jiang, D.H. Wang, Formation of singularities of solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics, Nonlinearity 23 (2010) 809–821.
- [5] T.P. Liu, Nonlinear resonance for quasilinear hyperbolic equations, J. Math. Phys. 28 (11) (1987) 2593–2602.
- [6] J. Hong, B. Temple, The generic solution of the Riemann problem in a neighborhood of a point of resonance for systems of nonlinear balance laws, Methods Appl. Anal. 10 (2003) 279–294.
- [7] J. Hong, B. Temple, A bound on the total variation of the conserved quantities for solutions of a general resonant nonlinear balance law, SIAM J. Appl. Math. 64 (2004) 819–857.
- [8] E. Isaacson, B. Temple, Convergence of the 2×2 Godunov method for a general resonant nonlinear balance law, SIAM J. Appl. Math. 55 (1995) 625–640.
- [9] T.P. Liu, Hyperbolic conservation laws with relaxation, Comm. Math. Phys. 108 (1987) 153–175.
- [10] C. Sinestrari, The Riemann problem for an inhomogeneous conservation laws without convexity, SIAM J. Math. Anal. 28 (1997) 109–135.
- [11] P.D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, SIAM, Philadelphia, 1973.
- [12] J. Smoller, Shock Waves and Reaction–Diffusion Equations, second ed., Springer-Verlag, New York, 1994.
- [13] A. Bressan, Hyperbolic systems of Conservation Laws: The One-Dimensional Cauchy Problem, second ed., Oxford University Press, Oxford, New York, 2000.
- [14] S.N. Kruzhkov, First order quasilinear equations with several space variables, Math. USSR Sb. 10 (1970) 217–243.